135. On the Dimension of Product Spaces

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The purpose of the present note is to give a sufficient condition under which the inequality $\operatorname{Ind} R\times S\leq \operatorname{Ind} R+\operatorname{Ind} S$ holds good, where Ind denotes the large inductive dimension. We define inductively $\operatorname{Ind} R$. Let $\operatorname{Ind}\phi=-1$, where ϕ is the empty set. $\operatorname{Ind} R\leq n\ (=0, 1, 2,\cdots)$ if and only if for any pair $F\subset G$ of a closed set F and an open set G there exists an open set H with $F\subset H\subset G$ such that $\operatorname{Ind}(\bar{H}-H)\leq n-1$. When $\operatorname{Ind} R\leq n-1$ is false and $\operatorname{Ind} R\leq n$ is true, we call $\operatorname{Ind} R=n$. When $\operatorname{Ind} R\leq n$ is false for any n, we call $\operatorname{Ind} R=\infty$.

Let $\mathfrak{U}$ be a collection of subsets of a topological space R. Then we call $\mathfrak{U}$ is *discrete* or *locally finite* if every point of R has a neighborhood which meets at most respectively one element or finite elements of $\mathfrak{U}$. We call $\mathfrak{U}$ is *σ-discrete* or *σ-locally finite* if $\mathfrak{U}$ is a sum of a countable number of discrete or locally finite subcollections respectively. A *binary covering* is a covering which consists of two elements.

Lemma 1. *Let R be a hereditarily paracompact Hausdorff space. Then the following statements are valid.*

1) (Subset theorem). *For any subset T of R* $\operatorname{Ind} T\leq \operatorname{Ind} R$.

2) (Sum theorem). *If F_i, $i=1, 2,\cdots$, are closed,* $\operatorname{Ind}\bigcup_{i=1}^{\infty} F_i=\sup \operatorname{Ind} F_i$.

3) (Local dimension theorem). *For any collection $\mathfrak{U}$ of open sets* $\operatorname{Ind}\bigcup\{U; U\in\mathfrak{U}\}=\sup\{\operatorname{Ind} U; U\in\mathfrak{U}\}$.

This is proved by C. H. Dowker [1]. The main part of the following lemma is essentially proved in Morita [4], but we give here full proof for the sake of completeness.

Lemma 2. *In a hereditarily paracompact Hausdorff space R the following conditions are equivalent.*

1) $\operatorname{Ind} R\leq n$.

2) *Every open covering can be refined by a locally finite and σ-discrete open covering* $\mathfrak{V}$ such that for any $V\in\mathfrak{V}$ $\operatorname{Ind}(\bar{V}-V)\leq n-1$.

3) *Every binary open covering can be refined by a σ-locally finite open covering $\mathfrak{V}$ such that for any $V\in\mathfrak{V}$* $\operatorname{Ind}(\bar{V}-V)\leq n-1$.

Proof. First we prove the implication 1)$\rightarrow$2). Let $\mathfrak{U}$ be an arbitrary open covering of R; then by A. H. Stone's theorem [5] $\mathfrak{U}$

can be refined by an open covering $\bigcup_{i=1}^{\infty}\mathfrak{U}_i$, where each $\mathfrak{U}_i=\{U(i,\alpha);\ \alpha\in A_i\}$ is a discrete collection of open sets. Let $U_i=\bigcup\{U(i,\alpha);\ \alpha\in A_i\}$, $i=1,2,\cdots$; then $\{U_i;\ i=1,2,\cdots\}$ can be refined by a locally finite open covering $\{W_i;\ i=1,2,\cdots\}$ such that $W_i\subset U_i$ for every i. Since a paracompact Hausdorff space is normal and locally finite open covering of a normal space is shrinkable,[1] $\{W_i;\ i=1,2,\cdots\}$ can be refined by a closed covering $\{F_i;\ i=1,2,\cdots\}$ such that $F_i\subset W_i$ for every i. Let V_i be an open set with $F_i\subset V_i\subset W_i$ such that $\operatorname{Ind}(\bar{V}_i-V_i)\leq n-1$. Let $\mathfrak{V}_i=\{V(i,\alpha)=V_i\frown U(i,\alpha);\ \alpha\in A_i\}$; then $\mathfrak{V}=\bigcup_{i=1}^{\infty}\mathfrak{V}_i$ satisfies all the requirements in 2).

The implication 2)$\to$3) is evident.

Let us prove 3) implies 1). Let $F\subset G$ be an arbitrary pair of a closed set F and an open set G. Let L and M be open sets with $F\subset L\subset\bar{L}\subset M\subset\bar{M}\subset G$. The binary open covering $\{M, R-\bar{L}\}$ is refined by an open covering $\mathfrak{V}=\bigcup_{i=1}^{\infty}\mathfrak{V}_i$, where $\mathfrak{V}_i=\{V(i,\alpha);\ \alpha\in A_i\}$, $i=1,2,\cdots$, are locally finite, such that for any $V\in\mathfrak{V}$ $\operatorname{Ind}(\bar{V}-V)\leq n-1$. Let

(1) $$C_i=\bigcup\{\bar{V}-V;\ V\in\mathfrak{V}_i\},\quad C=\bigcup\{\bar{V}-V;\ V\in\mathfrak{V}\};$$

then we have $C=\bigcup_{i=1}^{\infty}C_i$. By Lemma 1 we have

(2) $$\operatorname{Ind}C\leq n-1.$$

Here we notice that by Lemma 1 $\operatorname{Ind}D\leq n-1$ for any subset D of C. Let

(3) $H_i=\bigcup\{V(i,\alpha);\ V(i,\alpha)\frown\bar{L}\neq\phi,\ \alpha\in A_i\}$, $K_i=\bigcup\{V(i,\alpha);\ V(i,\alpha)\frown\bar{L}=\phi,\ \alpha\in A_i\}$.

Put

(4) $$P_1=H_1,\ Q_1=K_1-\bar{H}_1,\ P_i=H_i-\bigcup_{j<i}\bar{K}_j,\ Q_i=K_i-\bigcup_{j\leq i}\bar{H}_j,\ i=2,3,\cdots,$$

(5) $$P=\bigcup_{i=1}^{\infty}P_i,\quad Q=\bigcup_{i=1}^{\infty}Q_i.$$

Then we have

(6) $$R=\bigcup_{i=1}^{\infty}\bar{P}_i\smile\Big(\bigcup_{i=1}^{\infty}\bar{Q}_i\Big),$$

(7) $$P\frown Q=\phi,\quad \bar{P}_i\subset\bar{M}\ (i=1,2,\cdots),\quad Q\frown\bar{L}=\phi.$$

Finally we put

(8) $$W=R-\bar{Q}.$$

Since $Q\frown L=\phi$ by (7) and L is open, we have $\bar{Q}\frown L=\phi$ and hence $F\subset L\subset V$. Since $V=R-\bar{Q}\subset R-\bigcup_{i=1}^{\infty}\bar{Q}_i\subset\bigcup_{i=1}^{\infty}\bar{P}_i\subset\bar{M}\subset G$ by (6) and (7), we have

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1) A covering $\{U_\alpha;\ \alpha\in A\}$ is called *shrinkable* if there exists a closed covering $\{F_\alpha;\ \alpha\in A\}$ such that $F_\alpha\subset U_\alpha$ for every $\alpha\in A$.

(9) $$F \subset W \subset G.$$

Since $\overline{P}_i = P_i \cup (\overline{P}_i - P_i)$ and $\overline{Q}_i = Q_i \cup (\overline{Q}_i - Q_i)$, we have from (6)

(10) $$R = P \cup Q \cup (\bigcup_{i=1}^{\infty} (\overline{P}_i - P_i)) \cup (\bigcup_{i=1}^{\infty} (\overline{Q}_i - Q_i)).$$

From (7) and the openness of P it follows that $P \frown \overline{Q} = \phi$. Hence $P \frown (\overline{Q} - Q) = \phi$. Therefore we have

(11) $$\overline{Q} - Q \subset \bigcup_{i=1}^{\infty} (\overline{P}_i - P_i) \cup (\bigcup_{i=1}^{\infty} (\overline{Q}_i - Q_i)).$$

Since $\overline{P}_i - P_i \subset \overline{H}_i - H_i$ by (4) and $\overline{H}_i - H_i \subset C_i \subset C$, we have

(12) $$\overline{P}_i - P_i \subset C.$$

Similarly we have

(13) $$\overline{Q}_i - Q_i \subset C.$$

Combining (12) and (13) with (11), we have $\overline{Q} - Q \subset C$ and hence

(14) $$\mathrm{Ind}\,(\overline{Q} - Q) \leq n-1.$$

Thus we have

(15) $$\mathrm{Ind}\,(\overline{W} - W) \leq n-1,$$

and the lemma is completely proved.

Lemma 3. *In a topological space R the following conditions are equivalent with each other.*

1) *R is a metrizable space with* $\mathrm{Ind}\, R \leq n$.

2) *There exists a σ-discrete open basis $\mathfrak{V}$ of R such that for every $V \in \mathfrak{V}$* $\mathrm{Ind}\,(\overline{V} - V) \leq n-1$.

3) *There exists a σ-locally finite open basis $\mathfrak{V}$ of R such that for every $V \in \mathfrak{V}$* $\mathrm{Ind}\,(\overline{V} - V) \leq n-1$.

Proof. The implication 2)$\rightarrow$3) is evident.

Let $\mathfrak{V}$ be a σ-locally finite open basis of R such that for every $V \in \mathfrak{V}$ $\mathrm{Ind}\,(\overline{V} - V) \leq n-1$. Then R is metrizable by a well-known meterization theorem of J. Nagata and Yu. M. Smirnov. Moreover we get $\mathrm{Ind}\, R \leq n$ by a theorem of Katětov [2] and Morita [4]. Hence 3) implies 1).

The implication 1)$\rightarrow$2) is verified as follows. Let R be a metric space with $\mathrm{Ind}\, R \leq n$. Then by Lemma 2 there exists for every positive integer i a σ-discrete open covering $\mathfrak{V}_i$ the diameter of each element of which is less than $1/i$ such that for every $V \in \mathfrak{V}_i$ $\mathrm{Ind}\,(\overline{V} - V) \leq n-1$. Then $\mathfrak{V} = \bigcup_{i=1}^{\infty} \mathfrak{V}_i$ is a σ-discrete open basis of R such that for every $V \in \mathfrak{V}$ $\mathrm{Ind}\,(\overline{V} - V) \leq n-1$, and the proof of the lemma is finished.

Lemma 4. *Let R be a perfectly normal,*[2] *paracompact space*

2) A space R is called *perfectly normal* if R is normal and every open subset of R is an F_σ.

and S a metric space. Then $R\times S$ is a hereditarily paracompact Hausdorff space.

This is proved by Michael [3].

Theorem. *Let R be a perfectly normal, paracompact space and S a metric space. If either $R\neq\phi$ or $S\neq\phi$ holds good, we have* $\operatorname{Ind} R\times S\leq \operatorname{Ind} R+\operatorname{Ind} S$.

Proof. $R\times S$ is hereditarily paracompact by Lemma 4. When $\operatorname{Ind} R$ or $\operatorname{Ind} S$ is infinite, the theorem trivially holds good. Hence we prove the theorem for the case $\operatorname{Ind} R=m<\infty$, $\operatorname{Ind} S=n<\infty$. We shall carry out the proof by the induction on $k=m+n$. When $m+n=-1$, either R or S is empty. Hence the theorem is evidently true. Now we assume that the theorem holds for the case when $\operatorname{Ind} R+\operatorname{Ind} S$ is smaller than k. Let $m+n=k$.

Let $\mathfrak{G}$ be an arbitrary binary open covering of $R\times S$. Let us construct a refinement of $\mathfrak{G}$ satisfying the condition 3) of Lemma 2. Let $\mathfrak{V}=\{V_\beta;\ \beta\in B=\bigcup_{i=1}^{\infty} B_i\}$ be an open basis of S such that for every $V_\beta\in\mathfrak{V}$ $\operatorname{Ind}(\overline{V}_\beta-V_\beta)\leq n-1$ and $\mathfrak{V}_i=\{V_\beta;\ \beta\in B_i\}$ is discrete for every i.

Let $\mathfrak{U}=\{U_\alpha;\ \alpha\in A\}$ be an open basis of R and

(16) $$C=\{(\alpha,\beta);\ (\alpha,\beta)\in A\times B,\ U_\alpha\times V_\beta \text{ refines } \mathfrak{G}\}.$$

Then evidently $\{U_\alpha\times V_\beta;\ (\alpha,\beta)\in C\}$ is an open covering of $R\times S$ which refines $\mathfrak{G}$. Let

(17) $$A_\beta=\{\alpha;\ (\alpha,\beta)\in C\},$$

and

(18) $$U_\beta=\bigcup\{U_\alpha;\ \alpha\in A_\beta\}.$$

Since R is perfectly normal, there exists a sequence of open sets $G_{\beta i}$, $i=1,2,\cdots$, such that

(19) $$\overline{G}_{\beta 1}\subset G_{\beta 2}\subset\overline{G}_{\beta 2}\subset G_{\beta 3}\subset\cdots \quad\text{and}\quad \bigcup_{i=1}^{\infty} G_{\beta i}=U_\beta.$$

Consider an open covering

(20) $$\mathfrak{U}_\beta=\{U_\alpha\cap G_{\beta i};\ \alpha\in A_\beta,\ i=1,2,\cdots\}$$

of U_β. Then by Lemmas 1 and 2 $\mathfrak{U}_\beta$ can be refined by an open covering $\mathfrak{W}_\beta=\bigcup_{i=1}^{\infty}\mathfrak{W}_{\beta i}$ of U_β, where each $\mathfrak{W}_{\beta i}$ is discrete in U_β, such that for every $W\in\mathfrak{W}_\beta$ $\operatorname{Ind}(\overline{W}-W)\leq n-1$. Here we notice that the closure of $W\in\mathfrak{W}_\beta$ in U_β is the same as that in the whole space R by (19). Let

(21) $$\mathfrak{W}_{\beta ij}=\{W;\ W\in\mathfrak{W}_{\beta i},\ W\subset G_{\beta j}\}.$$

Then $\mathfrak{W}_{\beta ij}$ is discrete in R by (19). Let

(22) $$\mathfrak{L}_{ijk}=\{W\times V_\beta;\ W\in\mathfrak{W}_{\beta ij},\ \beta\in B_k\}.$$

Then $\mathfrak{L}_{ijk}$ is discrete in $R\times S$. Since $\overline{W\times V_\beta}-W\times V_\beta=((\overline{W}-W)\times\overline{V}_\beta)\cup(\overline{W}\times(\overline{V}_\beta-V_\beta))$, we have

(23) $$\operatorname{Ind}(\overline{W\times V_\beta}-W\times V_\beta)\leq m+n-1,$$

for any $W\times V_\beta\in\mathfrak{L}_{ijk}$, by the induction assumption and Lemma 1. Evidently

$$\mathfrak{L}=\bigcup_{i,j,k=1}^{\infty}\mathfrak{L}_{ijk} \tag{24}$$

is an open covering of $R\times S$ and refines $\mathfrak{G}$. Thus we conclude that $\operatorname{Ind} R\times S\leq m+n$ by Lemma 2 and the theorem is proved.

References

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