133. General Analyses of the Forced Oscillations in a Waveguide and in a Cavity

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The electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ in a homogeneous and isotropic medium with medium constants ε, μ and σ, satisfy the equations

$$(1) \qquad \nabla\times\boldsymbol{E}=-i\omega\mu\boldsymbol{H}-\boldsymbol{k}_H, \quad \nabla\times\boldsymbol{H}=(\sigma+i\omega\varepsilon)\boldsymbol{E}+\boldsymbol{k}_E$$

when known distributions of densities of electromagnetic currents $\boldsymbol{k}_E$ and $\boldsymbol{k}_H$ are given. In the other paper,[1] the author has shown that the fields in anisotropic inhomogeneous medium, when discussed along the line of the theory of perturbation, satisfy the same equations as (1), in which $\boldsymbol{k}_E$ and $\boldsymbol{k}_H$ have been the perturbed terms. Therefore the analysis of (1) is not only interesting in itself but useful for studies of the fields in anisotropic inhomogeneous medium.

1. Suppose that $\boldsymbol{i}_z$ is the unit vector along z-axis, and that $\boldsymbol{E}=\boldsymbol{E}_t+\boldsymbol{i}_zE_z$, $\boldsymbol{H}=\boldsymbol{H}_t+\boldsymbol{i}_zH_z$, $\boldsymbol{k}_E=\boldsymbol{k}_{Et}+\boldsymbol{i}_zk_{Ez}$, $\boldsymbol{k}_H=\boldsymbol{k}_{Ht}+\boldsymbol{i}_zk_{Hz}$ and $\nabla=\nabla_t+\boldsymbol{i}_z\,\partial/\partial z$, then (1) will be reduced to

$$(2) \qquad \begin{aligned} &\partial\boldsymbol{E}_t/\partial z=\nabla_tE_z+i\omega\mu\boldsymbol{i}_z\times\boldsymbol{H}_t+\boldsymbol{i}_z\times\boldsymbol{k}_{Ht}, \quad i\omega\mu H_z=\nabla_t\cdot[\boldsymbol{i}_z\times\boldsymbol{E}_t]-k_{Hz} \\ &\partial\boldsymbol{H}_t/\partial z=\nabla_tH_z-(\sigma+i\omega\varepsilon)\boldsymbol{i}_z\times\boldsymbol{E}_t-\boldsymbol{i}_z\times\boldsymbol{k}_{Et}, \quad (\sigma+i\omega\varepsilon)E_z=-\nabla_t\cdot[\boldsymbol{i}_z\times\boldsymbol{H}_t]-k_{Ez}. \end{aligned}$$

To begin with, we shall consider about the fields in a waveguide, the axis of which is parallel to z axis. Let $\mathfrak{L}$ be the two sided Laplace transformation defined as $\mathfrak{L}\{\boldsymbol{F}(z)\}=\boldsymbol{F}(s)=s\int_{-\infty}^{\infty}e^{-sz}\boldsymbol{F}(z)dz$. Applying $\mathfrak{L}$ to (2), we shall have, after some calculations,

$$(3) \qquad \begin{aligned} &k^2\boldsymbol{E}_t=s\nabla_tE_z+i\omega\mu\boldsymbol{i}_z\times\nabla_tH_z+i\omega\mu\boldsymbol{k}_{Et}+s\boldsymbol{i}_z\times\boldsymbol{k}_{Ht} \\ &k^2\boldsymbol{H}_t=s\nabla_tH_z-(\sigma+i\omega\varepsilon)\boldsymbol{i}_z\times\nabla_tE_z+(\sigma+i\omega\varepsilon)\boldsymbol{k}_{Ht}-s\boldsymbol{i}_z\times\boldsymbol{k}_{Et} \end{aligned}$$

$$(4) \qquad \Delta_tE_z+k^2E_z=g_E, \quad \Delta_tH_z+k^2H_z=g_H,$$

where $\kappa^2=-i\omega\mu(\sigma+i\omega\varepsilon)$, $k^2=\kappa^2+s^2$, $\Delta_t=\nabla_t\cdot\nabla_t$ and $g_E=-s\nabla_t\cdot\boldsymbol{k}_{Et}/(\sigma+i\omega\varepsilon)-\nabla_t\cdot[\boldsymbol{i}_z\times\boldsymbol{k}_{Ht}]-k^2k_{Ez}/(\sigma+i\omega\varepsilon)$, $g_H=-s\nabla_t\cdot\boldsymbol{k}_{Ht}/i\omega\mu+\nabla_t\cdot[\boldsymbol{i}_z\times\boldsymbol{k}_{Et}]-k^2k_{Hz}/i\omega\mu$. Conversely, assume that E_z and H_z satisfy (4), and that $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$ be defined by the right hand sides of (3) with the solutions E_z and H_z of (4), then it is easy to see that these values of $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy (2), that is (1). Therefore (1) is equivalent to (3) and (4).

Next we shall consider the fields in a cavity. Here a cavity means a finite domain which is enclosed by a waveguide and two planes of conductor perpendicular to the axis of the waveguide. Let these planes be $z=0$ and $z=L$. Suppose that the finite Fourier sine and cosine transformations of $\theta(z)$, which is a function defined in

$0 \leqq z \leqq L$ and satisfies the Dirichlet condition there, are defined as $\theta_s(\nu)=\int_0^L \theta(z) \sin \gamma_\nu zdz$ and $\theta_c(\nu)=\int_0^L \theta(z) \cos\gamma_\nu zdz$ $(\gamma_\nu=\nu\pi/L)$ respectively. Applying the cosine transformation to the first and the last equations of (2), the sine transformation to the other equations of (2), and using the boundary conditions on $z=0$ and $z=L$, we shall have,

$$k_\nu^2 \boldsymbol{E}_{ts}(\nu)=-\gamma_\nu \nabla_t E_{zc}(\nu)+i\omega\mu \boldsymbol{i}_z \times \nabla_t H_{zs}(\nu)+i\omega\mu \boldsymbol{k}_{Ets}(\nu)-\gamma_\nu \boldsymbol{i}_z \times \boldsymbol{k}_{Htc}(\nu),$$

$$(5) \qquad k_\nu^2 \boldsymbol{H}_{tc}(\nu)=\gamma_\nu \nabla_t H_{zs}(\nu)-(\sigma+i\omega\varepsilon)\boldsymbol{i}_z \times \nabla_t E_{zc}(\nu) +(\sigma+i\omega\varepsilon)\boldsymbol{k}_{Htc}(\nu)-\gamma_\nu \boldsymbol{i}_z \times \boldsymbol{k}_{Etc}(\nu).$$

$$(6) \qquad \Delta_t H_{zs}(\nu)+k_\nu^2 H_{zs}(\nu)=g_{Hs}(\nu), \quad \Delta_t E_{zc}(\nu)+k_\nu^2 E_{zc}(\nu)=g_{Ec}(\nu),$$

where $k_\nu^2=\kappa^2-\gamma_\nu^2$, and the quantities with suffix s, such as $\boldsymbol{E}_{ts}$, represent the sine transformations, and the quantities with suffix c, such as $\boldsymbol{H}_{tc}$, represent the cosine transformations, of the corresponding field components.

Now, (5) and (6) will be reduced to (3) and (4) respectively, if the notations in the former, for instance k_ν, $\boldsymbol{E}_{ts}$, $\boldsymbol{H}_{tc}$, etc., are replaced by k, $\boldsymbol{E}_t$, $\boldsymbol{H}_t$, etc. Hence the analyses of the former will be available directly for the latter, and therefore we shall study about the analyses of (3) and (4) in the following.

2. In order to obtain the fields in a waveguide, it is necessary and sufficient to have the solutions E_z and H_z of (4) which satisfy the boundary conditions. Let C be a curve or curves, which are the intersections of a plane $z=z$ with the walls of waveguide, and let S be a domain enclosed by C. It makes no difference whether S is simply connected or not. If the waveguide is filled with a single medium, E_z and H_z are continuous in S. In this case, E_z and H_z can be expanded as

$$(7) \qquad E_z=\sum_m \{a_m\varphi_m+b_m\psi_m\}, \quad H_z=\sum_m \{c_m\varphi_m+d_m\psi_m\},$$

where φ_m and ψ_m are eigen functions of $\Delta_t\varphi_m+\lambda_{1m}^2\varphi_m=0$ and $\Delta_t\psi_m+\lambda_{2m}^2\psi_m=0$ with eigen values λ_{1m} and λ_{2m} respectively. Moreover, it has been assumed that the boundary conditions $\varphi_m=0$ and $\partial\psi_m/\partial n=0$ on C are satisfied, where $\partial/\partial n$ denotes the differentiation along the outer normal on C. The reason why two kinds of function system have been employed is that the boundary conditions on S, i.e. $E_z=0$ and $\partial H_z/\partial n=-sk_{Hn}/i\omega\mu-k_{E\tau}$, must be satisfied, where k_{Hn} and $k_{E\tau}$ are normal and tangential components of $\boldsymbol{k}_{Ht}$ and $\boldsymbol{k}_{Et}$ on C respectively. It is easy to see that the following conditions are necessary and sufficient in order that (7) is the solutions of (4) which satisfy the boundary conditions on C.

$$\sum_m b_m\psi_m=0, \quad \sum_m c_m\partial\varphi_m/\partial n=-sk_{Hn}/i\omega\mu-k_{E\tau}, \quad \text{on } C,$$

$$(8) \qquad a_m=1/(h_{1m}^2-s^2)\left\{\sum_\nu (s^2-h_{2\nu}^2)b_\nu\int_S \varphi_m\psi_\nu dS-\int_S g_E\varphi_m dS\right\}$$

$$d_m=1/(h_{2m}^2-s^2)\left\{\sum_\nu (s^2-h_{1\nu}^2)c_\nu\int_S \varphi_\nu\psi_m dS-\int_S g_H\psi_m dS\right\},$$

where $h_{jm}^2=\lambda_{jm}^2-\kappa^2$, $(j=1,2)$.
When the solutions a_m, b_m, c_m and d_m of (8) are obtained, E_z and H_z, determined by (7), and $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$, determined by (3), are the desired field components of the forced oscillation in a waveguide. Because of the linearity of (8), it is easy to see that these solutions are unique up to the field components of free oscillation in the waveguide. This ambiguity will be removed when the total fields on one plane $z=z$ are given. Thus we have had the formulae to give the fields in a waveguide completely.

For an example, we shall study about the forced oscillation in a circular waveguide of radius a, in the presence of the unit radial magnetic current density. Without loss of generality, the coordinates of the current may be $r=\rho$, $\theta=0$ and $z=0$ $(\rho<a)$ in the cylindrical coordinates system. Since $\boldsymbol{k}_E=0$ and $\boldsymbol{k}_H=\boldsymbol{i}_r\delta(r-\rho)\delta(\theta)\delta(z)$, we have $\boldsymbol{k}_E(s)=0$ and $\boldsymbol{k}_H(s)=\boldsymbol{i}_r s\delta(r-\rho)\delta(\theta)$. Substituting these values into (8), it will be able to calculate the values of a_m etc., with which we shall have $E_z=-\sum_{m,n} s\Big/(h_{1mn}^2-s^2)imA_{mn}^2J_m(\lambda_{1mn}\rho)J_m(\lambda_{1mn}r)e^{im\theta}$, where J_m is the Bessel function of m-th order and A_{mn} is the normalizing factor of $J_m(\lambda_{1mn}r)e^{im\theta}$. All other components of $\boldsymbol{E}$ and $\boldsymbol{H}$ can be obtained, but it will be omitted to describe for the sake of brevity. Difficulties remain still in the calculations of the inverse Laplace transformation $\mathfrak{L}^{-1}$, but in this case, we can compute the inverse transformations of $E_z(s)$ etc. obtained above, by virtue of the formulae $\mathfrak{L}^{-1}\{hs/h^2-s^2\}=1/2\cdot e^{-h|z|}$, $\mathfrak{L}^{-1}\{f(s)/s\}=\int_{-\infty}^{z}h(\zeta)d\zeta$ and $\mathfrak{L}^{-1}\{sf(s)\}=\dfrac{d}{dz}h(z)(Re\, s>0)$, where $f(s)=\mathfrak{L}\{h(z)\}$. We shall be able to find that the 'strip of convergence' in our example is $0<Re\, s<Re\, i\mathring{\kappa}$, where $\mathring{\kappa}$ is one of the roots of κ^2, of which $Im\,\mathring{\kappa}<0$. Thus we shall have the desired field component $E_z=\sum_{m,n}(-im/2\mathring{h}_{1mn})A_{mn}^2J_m(\lambda_{1mn}\rho)J_m(\lambda_{1mn}r)e^{im\theta-\mathring{h}_{1mn}|z|}$, where $\mathring{h}_{1mn}$ is one of the roots of $h_{1mn}^2=\lambda_{1mn}^2-\kappa^2$, of which $Re\,\mathring{h}_{1mn}>0$. Descriptions of other components of the fields have also been omitted, for the sake of simplicity.

3. Next we shall study about the plural numbers of media. It is sufficient to consider of two distinct media, since similar holds for more numbers of media.

Let K be a curve which lies in the interior of S. Suppose that S_i is the domain bounded by K, and S_e is the domain bounded by K and C. $(S=S_i+S_e)$. Quantities, such as the fields, forced terms, and medium constants, will be indicated by suffix i or e that to which medium they belong. Then the boundary conditions on K will be; $E_{iz}=E_{ez}$ (is put $=E_z$), $H_{iz}=H_{ez}(=H_z)$, $\partial E_z/\partial\tau=1/\Gamma_i\,\partial H_{iz}/\partial n-1/\Gamma_e\,\partial H_{ez}/\partial n+h_E$, and $\partial H_z/\partial\tau=1/\Lambda_i\,\partial E_{iz}/\partial n-1/\Lambda_e\,\partial E_{ez}/\partial n+h_H$; where $\partial/\partial\tau$ denotes the tangential differentiation on K, and h_E and h_H are known functions constructed with $\boldsymbol{k}_E$ and $\boldsymbol{k}_H$ in S_i and S_e. Also Γ_j and Λ_j $(j=i, e)$ are constants defined as $1/\Gamma_j = -i\omega\mu_j/s(\kappa_e^2-\kappa_i^2)\cdot(k_ik_e/k_j)^2$ and $\Lambda_j = -i\omega\mu_j/(\sigma_j+i\omega\varepsilon_j)\cdot\Gamma_j$ where $k_j^2=\kappa_j^2+s^2$. On the other hand, the boundary conditions on C are $E_{ez}=0$ and $\partial H_{ez}/\partial n = -sk_{eHn}/i\omega\mu_e - k_{eE\tau}$. Our task is to find the solutions E_j and H_j of $\Delta_t E_{jz}+k_j^2E_{jz}=g_{jE}$ and $\Delta_t H_{jz}+k_j^2H_{jz}=g_{jH}$ which satisfy the boundary conditions on K and C.

Suppose $G(P,Q)=1/4_i\cdot H_0^{(2)}(k_eR)$, where R is the distance of two points P and Q; $R=\overline{PQ}$, and $H_0^{(2)}$ is the Hankel function of the second kind of 0-th order. Let G_n and G_τ be the normal and tangential derivatives of G on C and K. Then, by the help of Green's formula and the boundary conditions mentioned above, E_z and H_z will be represented as follows:

$$
(9)\quad
\begin{aligned}
E_{iz}(P_i) &= (k_i^2-k_e^2)\int_{S_i} G(P_iQ)E_{iz}(Q)dS_Q - \left(1-\frac{\Lambda_i}{\Lambda_e}\right)\int_K G_n(P_iQ)E_z(Q)dS_Q \\
&\quad -\Lambda_i\int_C G_\tau(P_iQ)H_z(Q)ds_Q + \frac{\Lambda_i}{\Lambda_e}\int_C G(P_iQ)\partial E_{ez}/\partial n\, ds_Q - f_{iE}(P_i) \\
H_{iz}(P_i) &= (k_i^2-k_e^2)\int_{S_i} G(P_iQ)H_{iz}(Q)dS_Q - \left(1-\frac{\Gamma_i}{\Gamma_e}\right)\int_K G_n(P_iQ)H_z(Q)ds_Q \\
&\quad -\Gamma_i\int_C G_\tau(P_iQ)E_z(Q)ds_Q - \frac{\Gamma_i}{\Gamma_e}\int_C G_n(P_iQ)H_{ez}(Q)ds_Q - f_{iH}(P_i)
\end{aligned}
$$

$$
(10)\quad
\begin{aligned}
E_{ez}(P_e) &= \frac{\Lambda_e}{\Lambda_i}(k_i^2-k_e^2)\int_{S_i} G(P_eQ)E_{iz}(Q)dS_Q + \left(1-\frac{\Lambda_e}{\Lambda_i}\right)\int_K G_n(P_eQ)E_z(Q)ds_Q \\
&\quad -\Lambda_e\int_K G_\tau(P_eQ)H_z(Q)ds_Q + \int_C G(P_eQ)\partial E_{ez}/\partial n\, ds_Q - f_{eE}(P_e) \\
H_{ez}(P_e) &= \frac{\Gamma_e}{\Gamma_i}(k_i^2-k_e^2)\int_{S_i} G(P_eQ)H_{iz}(Q)dS_Q + \left(1-\frac{\Gamma_e}{\Gamma_i}\right)\int_K G_n(P_eQ)H_z(Q)ds_Q \\
&\quad -\Gamma_e\int_K G_\tau(P_eQ)E_z(Q)ds_Q - \int_C G_n(P_eQ)H_{ez}(Q)ds_Q - f_{eH}(P_e)
\end{aligned}
$$

where P_i and P_e are points which belong to S_i and S_e respectively. Also the last terms f in (9) and (10) are the known functions which are constructed with $\boldsymbol{k}_E$ and $\boldsymbol{k}_H$, the details of which have been omitted to describe here. Thus the fields in S_i and S_e have been represented by the integrals in the right hand sides of (9) and (10) respectively. Since these integrals contain the values of E_z and H_z in S_i, E_z and H_z on K, and $\partial E_{ez}/\partial n$ and H_{ez} on C, we must have the integral equations to determine these unknown quantities. Taking the limits of (9) as P_i tends to a point P on K, and limits of (10) as P_e tends to a point P on C, and using the results which have been obtained concerning to the limits of logarithmic simple and double layer potentials, we shall have the desired system of integral equations of Fredholm type, after eliminating the singular integrals appearing there by the help of the Poincaré-Bertrand formula, as follows:

$$\frac{1}{2}\left\{\frac{\Lambda_e+\Lambda_i}{\Lambda_e}+\frac{\Lambda_i\Gamma_i\Gamma_e}{\Gamma_e+\Gamma_i}\right\}E_z(P)$$
$$=(k_i^2-k_e^2)\int_{Si}\left\{G(P,Q)E_z(Q)-\frac{2\Lambda_i\Gamma_e}{\Gamma_e+\Gamma_i}G_{\tau 0}(PQ)H_z(Q)\right\}dS_Q$$
$$+\int_K\left\{\frac{2\Lambda_i\Gamma_i\Gamma_e}{\Gamma_e+\Gamma_i}G_{\tau\tau}(PQ)-\frac{\Lambda_e-\Lambda_i}{\Lambda_e}G_n(PQ)\right\}E_z(Q)ds_Q$$
$$+\int_K\frac{2\Lambda_i(\Gamma_e-\Gamma_i)}{\Gamma_e+\Gamma_i}G_{\tau n}(PQ)H_z(Q)ds_Q+\int_C\left\{\frac{2\Lambda_i\Gamma_i}{\Gamma_e+\Gamma_i}G_{\tau n}(PQ)H_{ez}(Q)\right.$$
$$\left.+\frac{\Lambda_i}{\Lambda_e}G(PQ)\partial E_{ez}(Q)/\partial n\right\}ds_Q+\frac{2\Gamma_e}{\Gamma_e+\Gamma_i}\int_C G(PQ)\partial f_{iH}(Q)/\partial\tau ds_Q-f_{iE}(P).$$

(11)
$$\frac{1}{2}\left\{\frac{\Gamma_e+\Gamma_i}{\Gamma_e}+\frac{\Gamma_i\Lambda_i\Lambda_e}{\Lambda_e+\Lambda_i}\right\}H_z(P)$$
$$=(k_i^2-k_e^2)\int_{Si}\left\{G(PQ)H_z(Q)-\frac{2\Lambda_e\Gamma_i}{\Lambda_e+\Lambda_i}G_{\tau 0}(PQ)E_z(Q)\right\}dS_Q$$
$$+\int_K\left\{\frac{2\Gamma_i\Lambda_i\Lambda_e}{\Lambda_e+\Lambda_i}G_{\tau\tau}(PQ)-\frac{\Gamma_e\Gamma_i}{\Gamma_e}G_n(PQ)\right\}H_z(Q)ds_Q$$
$$+\int_K\frac{2\Gamma_i(\Lambda_e-\Lambda_i)}{\Lambda_e+\Lambda_i}G_{\tau n}(PQ)E_z(Q)ds_Q-\int_C\left\{\frac{2\Gamma_i\Lambda_i}{\Lambda_e+\Lambda_i}G_{\tau 0}(PQ)\partial E_{ez}/\partial n\right.$$
$$\left.+\frac{\Gamma_i}{\Gamma_e}G_n(PQ)H_{ez}(Q)\right\}ds_Q+\frac{2\Lambda_e}{\Lambda_e+\Lambda_i}\int_K G(PQ)\partial f_{iE}(Q)/\partial n\, ds_Q-f_{iH}(P),$$
$$(P\in K).$$

$$\frac{1}{2}\partial E_{ez}(P)/\partial n=\int_C\partial G(PQ)/\partial n_P\cdot\partial E_{ez}(Q)/\partial n_Q ds_Q$$
$$+\frac{\Lambda_e}{\Lambda_i}(k_i^2-k_e^2)\int_{Si}\partial G(PQ)/\partial n_P\cdot E_{iz}(Q)ds_Q$$
$$+\left(1-\frac{\Lambda_e}{\Lambda_i}\right)\int_K\partial^2 G(PQ)/\partial n_P\partial n_Q\cdot E_z(Q)ds_Q$$

(12)
$$-\Lambda_e\int_K\partial^2 G(PQ)/\partial n_P\partial\tau\cdot H_z(Q)ds_Q-\partial f_{eE}(P)/\partial n_P$$
$$\frac{1}{2}H_{ez}(P)=-\int_C G_n(PQ)H_{ez}(Q)ds_Q+\frac{\Gamma_e}{\Gamma_i}(k_i^2-k_e^2)\int_{Si}G(PQ)H_{iz}(Q)dS_Q$$
$$+\left(1-\frac{\Gamma_e}{\Gamma_i}\right)\int_K G_n(PQ)H_z(Q)ds_Q-\Gamma_e\int_K G_\tau(PQ)E_z(Q)ds_Q-f_{eH}(P),$$
$$(P\in C)$$

where $G_{\tau 0}(PQ)=\int_K^* G_\tau(PQ_1)G(Q_1Q)ds_{Q_1}$, $G_{\tau n}(PQ)=\int_K^* G_\tau(PQ_1)G_n(Q_1Q)ds_{Q_1}$ and $G_{\tau\tau}(PQ)=\int_K^* G_\tau(PQ_1)G_\tau(Q_1Q)ds_{Q_1}$.

Reference

1) Yoshio Hayashi: Perturbation theory of the electromagnetic fields in anisotropic inhomogeneous media, Proc. Japan Acad., **36**, 547 (1960).