130. On the Covering Theorem of Vitali

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Vitali's covering theorem, though of fundamental importance to the theory of derivation of set-functions and interval-functions in a Euclidean space of any number of dimensions, has the defect that its range of applicability is restricted to the case of the usual Lebesgue measure. In the present note the author wants to alleviate this shortcoming by extending the theorem, in one-dimension at least, to the case of outer Carathéodory measures.

The proof for this given below is essentially a modification of that for Vitali's theorem, due to S. Banach and expounded on pp. 109–111 of the *Theory of the Integral* by S. Saks (this treatise will be quoted simply as Saks in the sequel). The gist of our argument consists in a simple lemma which reduces to an evident assertion in the case of Lebesgue measure.

Our extension of Vitali's theorem would enable us, as in Saks, to derive a number of consequent theorems with the help of the usual techniques of real function theory, though space limitation prevents us from dwelling upon this matter; among other things we could obtain analogues of not a few of the results contained in the fourth chapter of Saks. The theory of relative derivation of set-functions, thus established, would then be useful to deal with the curvature of parametric curves of certain general type.

By an *outer measure* we shall understand in what follows an arbitrary outer Carathéodory measure in the Saks sense, defined on the class of all subsets of the real line and vanishing for the void set. The following lemma will be the kernel of the proof of our theorem.

Lemma. *Suppose that Γ is an outer measure and that I_0 is a linear closed interval containing another closed interval $I=[a,b]$. Given any real number $\lambda \geqq \Gamma(I)$, let us denote by M the join of all the closed intervals J such that*

(1) $$J \subset I_0, \quad IJ \text{ nonvoid}, \quad \Gamma(J) \leqq \lambda.$$

Then M is an interval (closed, open, or half-open) contained in I_0 and we have

$$\Gamma(M) \leqq \Gamma(I) + 2\lambda.$$

Proof. Clearly I is one of the intervals J. Thus M is a connected infinite subset of the real line and so an interval. Write now $A = M \cdot (-\infty, a)$ and $B = M \cdot (b, \infty)$, so that $M = I \smile A \smile B$ and hence

$\Gamma(M) \leqq \Gamma(I) + \Gamma(A) + \Gamma(B)$. It thus suffices to show that $\Gamma(A) \leqq \lambda$ and $\Gamma(B) \leqq \lambda$, say the former inequality by symmetry. We may clearly assume A nonvoid. For each point t of A there is, by hypothesis, a closed interval J containing t and fulfilling the conditions (1). Then $[t, a] \subset J$, and hence $[t, a) \subset A$ as well as $\Gamma([t, a)) \leqq \Gamma(J) \leqq \lambda$. This being so, let us denote by c the infimum of the set A. We have two cases to distinguish: If $c \in A$, then $A = [c, a)$ and we are plainly at an end. Otherwise we must have $A = (c, a)$, and extracting from A a non-increasing sequence $c_1, c_2, \cdots$ tending to the point c we readily find that $\Gamma(A) = \lim_n \Gamma([c_n, a)) \leqq \lambda$. This completes the proof.

Theorem. *Let Γ be an outer measure which assumes finite values for bounded sets. If a nonvoid family $\mathfrak{M}$ of linear closed intervals covers in the Vitali sense a set E, i.e. if for each point t of E there are in the family $\mathfrak{M}$ indefinitely short intervals containing t, then we can extract from $\mathfrak{M}$ a disjoint (finite or infinite) sequence of intervals $I_1, I_2, \cdots$ covering E almost entirely (Γ). That is, we have*

$$\Gamma(E - \bigcup_n I_n) = 0.$$

Proof. a) We shall first prove the theorem in the special case in which (i) we have $\Gamma(I) > 0$ for every interval I of $\mathfrak{M}$ and further (ii) the set E is bounded, i.e. contained in some closed interval I_0. We may obviously assume that, in addition, all the intervals of $\mathfrak{M}$ are situated in I_0. Assuming that no finite sequence of intervals ($\mathfrak{M}$) fulfils the assertion, we shall show in the sequel that there must then exist an infinite sequence of intervals ($\mathfrak{M}$) conforming to the assertion.

We shall construct the required sequence $I_1, I_2, \cdots$ by induction. In the first place we choose for I_1 an arbitrary interval of $\mathfrak{M}$. When the first n intervals have already been defined so as to be mutually disjoint, we determine the next interval I_{n+1} by the following process. By hypothesis the set $E_n = E - I_1 - \cdots - I_n$ cannot be void and so there are in $\mathfrak{M}$ intervals H which intersect E_n but are disjoint from all the intervals $I_1, \cdots, I_n$. The supremum of $\Gamma(H)$ for all such H is a finite positive number, which we denote by δ_n. We take now for I_{n+1} any H which fulfils the condition $\Gamma(H) > 2^{-1}\delta_n$. This procedure can be continued indefinitely and yields us an infinite disjoint sequence $I_1, I_2, \cdots$ of intervals ($\mathfrak{M}$) such that $\sum_{n=1}^{\infty} \Gamma(I_n) \leqq \Gamma(I_0)$.

In order to see that the sequence $I_1, I_2, \cdots$ thus constructed covers E almost entirely (Γ), write $A = E - I_1 - I_2 - \cdots$ and suppose, if possible, that $\Gamma(A) > 0$. Denoting for each $n = 1, 2, \cdots$ by M_n the join of all closed intervals J subject to the conditions

$$J \subset I_0, \quad I_n J \text{ nonvoid}, \quad \Gamma(J) \leqq 2\Gamma(I_n),$$

we find at once, in view of the lemma, that M_n is an interval containing I_n and such that $\Gamma(M_n) \leqq 5\Gamma(I_n)$. Consequently

$$\sum_{n=1}^{\infty}\Gamma(M_n)\leqq 5\sum_{n=1}^{\infty}\Gamma(I_n)\leqq 5\Gamma(I_0),$$

so that the leftmost series is convergent. There is therefore a positive integer N satisfying

$$\Gamma(\bigcup_{n>N}M_n)\leqq\sum_{n>N}\Gamma(M_n)<\Gamma(A),$$

and this implies in particular that there must exist a point t_0 of A belonging to none of the sets $M_n(n>N)$. We observe in passing that t_0 belongs to none of $I_1, I_2, \cdots$.

This being so, let us select from $\mathfrak{M}$, as we plainly may, an interval I containing t_0 and disjoint from the intervals $I_1, \cdots, I_N$. Then I must intersect at least one of the intervals $I_n\,(n>N)$. For otherwise we should obtain for each $n=1, 2, \cdots$ the relation

$$0<\Gamma(I)\leqq\delta_n<2\Gamma(I_{n+1})\leqq 2\Gamma(M_{n+1}),$$

which is contradictory to the fact that $\Gamma(M_n)\to 0$ as $n\to\infty$. Hence there exists a minimal integer $k>N$ for which I_kI is nonvoid. It follows immediately that I_nI is void for $n=1, 2, \cdots, k-1$ and that therefore $\Gamma(I)\leqq\delta_{k-1}$. On the other hand, the point t_0 cannot belong to M_k and so I is not contained in M_k. Accordingly we infer at once from the construction of the set M_k that $2\Gamma(I_k)<\Gamma(I)$ and hence that $2\Gamma(I_k)<\delta_{k-1}$. But the last inequality is obviously incompatible with the definition of I_k. We have thus established the theorem under the additional assumptions (i) and (ii).

b) We now proceed to prove our theorem, only subject to the additional hypothesis (ii). For this purpose we may assume as in part a) that all the intervals of $\mathfrak{M}$ lie in I_0. Let us define $\Phi(X)=\Gamma(X)+|X|$ for any set X of real numbers. Then Φ is an outer measure which assumes finite values for bounded sets and positive values for closed intervals. Consequently, in accordance with what has already been proved in part a), there exists in the family $\mathfrak{M}$ a disjoint sequence Δ of intervals such that, if we denote by K the join of all the intervals of Δ, then $\Gamma(E-K)\leqq\Phi(E-K)=0$. This proves the theorem under the assumption (ii).

c) We are now in a position to treat the general case. We begin by observing that for the set E under consideration and for any monotone ascending or descending sequence $X_1, X_2, \cdots$ of bounded Borel sets we have

$$\Gamma(E\cdot\lim_n X_n)=\lim_n\Gamma(EX_n), \tag{2}$$

which follows directly from the theorem on p. 46 of Saks.

Let us write $P_n=E\cdot[-n, n]$ for positive integers n. We proceed to define by induction an infinite sequence $\mathfrak{T}_1, \mathfrak{T}_2, \cdots$ of finite families of intervals ($\mathfrak{M}$) as follows. We choose for $\mathfrak{T}_1$ any family consisting of a single interval of $\mathfrak{M}$. When the finite family $\mathfrak{T}_n$ has already been defined, we write T_n for the join of the intervals of $\mathfrak{T}_n$ and distinguish two cases according as T_n contains P_{n+1} or not. In the former case we set simply $\mathfrak{T}_{n+1}=\mathfrak{T}_n$. In the latter case we infer from part b) and the relation (2) that there is a nonvoid, disjoint, finite family $\mathfrak{U}_n$ of intervals ($\mathfrak{M}$), such that

$$\Gamma(P_{n+1}-T_n-U_n)<(n+1)^{-1},$$

where U_n denotes the join of the intervals of $\mathfrak{U}_n$ and may plainly be supposed disjoint from T_n. We define now $\mathfrak{T}_{n+1}=\mathfrak{T}_n\cup\mathfrak{U}_n$.

Thus constructed, each family $\mathfrak{T}_n$ is finite and disjoint. Since $\mathfrak{T}_1\subset\mathfrak{T}_2\subset\cdots$, the join $\mathfrak{T}$ of all the families $\mathfrak{T}_n$ is likewise disjoint. Moreover $\Gamma(P_n-T_n)<n^{-1}$ for $n>1$ by construction. Consequently, writing T for the join of the intervals of $\mathfrak{T}$, we have *a fortiori* $\Gamma(P_n-T)>n^{-1}$ for $n>1$. From the last inequality we deduce by means of (2) that $\Gamma(E-T)=0$. In other words, the disjoint countable family $\mathfrak{T}$ of intervals ($\mathfrak{M}$) covers the given set E almost entirely (Γ). The theorem is thus completely proved.

Remark. Instead of considering a family of intervals we could as well have dealt in the above with one consisting of closed sets. We did not put this possibility into practice because it would have caused our argument to become somewhat cumbersome. Indeed we should have been obliged, in that case, to take into account the parameters of regularity (Γ) of the closed sets under consideration, as might be easily seen on consulting the Saks treatise for the treatment of the Vitali theorem.