129. *The Diffusion Satisfying Wentzell's Boundary Condition and the Markov Process on the Boundary. I*

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(Comm. by Z. SUETUNA, M.J.A., Nov. 12, 1960)

1. W. Feller [1, 2] determined all the diffusion processes in one dimension. He obtained the intrinsic form of the differential operator for the diffusion equation and constructed solutions for all the possible boundary conditions. As an approach to such a solution in multi-dimensional case, A. D. Wentzell [7] formulated the problem in the following way.[1] Let D be a bounded domain in an N-dimensional orientable manifold of class C^∞ with sufficiently smooth boundary ∂D. Given a diffusion equation

$$(1)\qquad \frac{\partial}{\partial t}u = Au,$$

$$(2)\qquad \begin{aligned} Au(x) = {} & a(x)^{-\frac{1}{2}}\sum_{i,j=1}^{N}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\,a(x)^{\frac{1}{2}}\frac{\partial u(x)}{\partial x^j}\right) \\ & -a(x)^{-\frac{1}{2}}\sum_{i,j=1}^{N}\frac{\partial}{\partial x^i}(b^i(x)\,a(x)^{\frac{1}{2}}u(x))+c(x)u(x), \end{aligned}$$

where $\{a^{ij}(x)\}$ and $\{b^i(x)\}$ are contravariant tensors on $\bar{D}$ of class C^3, $\{a^{ij}(x)\}$ is symmetric and positive definite for each $x\in\bar{D}$, $c(x)\in C^2(\bar{D})$ is non-positive and $a(x)=\det(a_{ij}(x))$ where $a_{ij}(x)$ is the conjugate covariant tensor of $(a^{ij}(x))$. $C^n(\bar{D})$ and $C^n(\partial D)$ are the spaces of n-times continuously differentiable functions on $\bar{D}$ and ∂D respectively. We write $C(\bar{D})$ and $C(\partial D)$ for $C^0(\bar{D})$ and $C^0(\partial D)$ respectively. $C^H(\bar{D})$ is the space of uniformly Hölder continuous functions on $\bar{D}$. The norms of these spaces are those of uniform convergence. Operator A, defined on $C^2(\bar{D})$, has the closure $\bar{A}$ in $C(\bar{D})$. Now, the problem is to *find all the semi-groups $\{T_t, t\geq 0\}$ of non-negative linear operators T_t on $C(\bar{D})$ with norm $\|T_t\|\leq 1$, strongly continuous in $t\geq 0$, and having a contraction of $\bar{A}$ as its generator $\mathfrak{G}$ in Hille-Yosida sense.* Wentzell proved that, for any $u\in C^2(\bar{D})\frown\mathfrak{D}(\mathfrak{G})$ and $x_0\in\partial D$ we have

$$(3)\qquad Lu(x_0)=0,\quad x_0\in\partial D,$$

$$(4)\qquad \begin{aligned} Lu(x) = {} & \sum_{i,j=1}^{N-1}\alpha^{ij}(x)\frac{\partial^2 u(x)}{\partial\xi_x^i\xi_x^j}+\sum_{i=1}^{N-1}\beta^i(x)\frac{\partial u(x)}{\partial\xi_x^i}+\gamma(x)u(x) \\ & +\delta(x)\lim_{y\to x}Au(y)+\mu(x)\frac{\partial u(x)}{\partial n}+\int_{\bar{D}}\left\{u(y)-u(x)-\sum_{i=1}^{N-1}\frac{\partial u(x)}{\partial\xi_x^i}\xi_x^i(y)\right\}\nu_x(dy), \end{aligned}$$

1) The following set up is slightly modified for our present use.

where $\{a^{ij}(x)\}$ is symmetric and non-negative definite, $\gamma(x)$, $\delta(x)$, $-\mu(x)$ are non-positive and $\nu_x(\cdot)$ is a measure on D satisfying $\nu_x(\bar{D}-U_x)<\infty$ and $\int_{\bar{D}}\left\{\sum_{i=1}^{N-1}(\xi_x^i(y))^2+\xi_x^N(y)\right\}\nu_x(dy)<\infty$ for any neighbourhood U_x of x. $\nu_x(D)>0$ if all other terms in L are zero. When ∂D is represented by $\psi(x)=0$ in U_x and $\psi(x)>0$ in $U_x\frown D$, we understand $\frac{\partial u}{\partial n}(x)$ $=\sum_{i,j=1}^{N}a^{ij}(x)\frac{\partial u(x)}{\partial x^i}\frac{\partial u(x)}{\partial x^j}\cdot\tilde{\psi}(x)^{-1}$, where $\tilde{\psi}(x)=\left(\sum_{i,j=1}^{N}a^{ij}(x)\frac{\partial\psi(x)}{\partial x^i}\frac{\partial\psi(x)}{\partial x^j}\right)^{\frac{1}{2}}$. $\{\xi_x^i(y),\ 1\leq i\leq N\}$ is a system of $C^2(\bar{D})$ functions and is a local coordinate in U_x satisfying: $\xi_x^N(y)=0$ if and only if $y\in\partial D$ for $y\in U_x$, $\xi_x^N(y)$ $+\sum_{i=1}^{N-1}(\xi_x^i(y))^2>0$ if and only if $y\neq x$. Moreover $\frac{\partial}{\partial n}u(x)=\frac{\partial u(x)}{\partial\xi_x^N}$ holds. We sometimes omit the suffix x of $\xi_x^i(y)$. Wentzell also proved that all the diffusions in a solid sphere in R^3 or a disc in R^2 are characterized by the conditions of type (4).

The purpose of the present note is to describe, without proof, sufficient conditions that (1) and (3) have a solution. A detailed proof will be published elsewhere. The method is an extension of Feller's idea in one dimensional case and is entirely different from that of Wentzell. *We assume* i) *$Lu(x)$ is continuous in $x\in\partial D$ for each $u\in C^2(\bar{D})$,* ii) *$\nu_x(D)=\infty$ if $\delta(x)=\mu(x)=0$, for each $x\in\partial D$.*

2. *Semi-groups on $C(\partial D)$.* It can be proved that the equation $(\alpha-A)u(x)=0$ for $x\in D$, $\alpha\geq 0$ and $u(x)=f(x)$ for $x\in\partial D$, $f\in C(\partial D)$ has the unique solution which is represented by

$$u(x)=H_\alpha f(x)=\int_{\partial D}H_\alpha(x,dy)f(y),$$

where $H_\alpha(x,\cdot)$ is a measure on ∂D with $H_\alpha(x,\partial D)\leq 1$. H_α as a mapping from $C(\partial D)$ into $C(\bar{D})$, gives an isomorphism between $C(\partial D)$ and the Banach space $C_\alpha=\{u\in C(\bar{D})\,|\,(\alpha-A)u(x)=0,\ x\in D\}$. Moreover H_α maps $C^3(\partial D)$ into $C^2(\bar{D})$. Define

$$LH_\alpha:\qquad (LH_\alpha)f=L(H_\alpha f),$$

for such an $f\in C(\partial D)$ that $L(H_\alpha f)\in C(\partial D)$, and write $\tilde{L}$ and $\widetilde{LH}_\alpha$ for the restrictions of L and LH_α on the spaces $\mathfrak{D}(\tilde{L})=\{u\in C^H(\bar{D})\,|\,Lu\in C(\partial D)\}$ and $\mathfrak{D}(\widetilde{LH}_\alpha)=\{f\in C^H(\partial D)\,|\,H_\alpha f\in\mathfrak{D}(\tilde{L})\}$ respectively. $\tilde{L}$, considered as a mapping from the subspace $\mathfrak{D}(\tilde{L})\frown C_\alpha$ of C_α into $C(\partial D)$, has the closure L_α. $\widetilde{LH}_\alpha$ also has the closure $\overline{LH}_\alpha$. We have

Theorem 1. *Fix an $\alpha\geq 0$. Assume that there is a dense subspace $\mathfrak{D}_\alpha$ of $C(\partial D)$ such that* i) *the equation*

(5) $\qquad(\alpha-A)u(x)=0,\ x\in D$ *and* $(\beta-L)u(x)=f(x),\ x\in\partial D$

has a solution $u \in C^H(\partial D)$ *for each* $\beta > 0$ *and* $f \in \mathfrak{D}_\alpha$, *or more* generally,
ii) the equation

$$(\beta - \overline{LH_\alpha})g = f \tag{6}$$

has a solution $g \in C^H(\partial D)$ *for each* $\beta > 0$ *and* $f \in \mathfrak{D}_\alpha$. *Then, there is a unique semi-group* $\{T_t^\alpha\}$ *of non-negative linear operators* T_t^α *on* $C(\partial D)$, *with norm* $\|T_t^\alpha\| \le 1$, *strongly continuous in* $t \ge 0$, *with the generator* $\overline{LH_\alpha}$. *The solutions of* (5) *and* (6) *are unique and represented by* $u = H_\alpha K_\beta^\alpha f$ *and* $g = K_\beta^\alpha f$ *respectively, where* K_β^α *is the resolvent for* $\{T_t^\alpha\}$ *with parameter* $\beta > 0$. (Hence, there is a Markov process on ∂D with the generator $\overline{LH_\alpha}$.)

Noting that $LH_\alpha 1(x) < 0$ for $x \in \partial D$ and $\alpha > 0$ by the assumption ii) on L, and hence considering a slightly modified semi-group on $C(\partial D)$, we have

Corollary. *Under the condition of Theorem* 1 *the equation*

$$\overline{LH_\alpha} g = f, \quad \alpha > 0$$

has a unique solution $g \in \mathfrak{D}(\overline{LH_\alpha})$ *for each* $f \in C(\partial D)$.

3. *Resolvent operators* $\{G_\alpha\}$. The Green function $g_\alpha(x, y)$, $\alpha \ge 0$ for the equation

$$(\alpha - A)u(x) = 0, \ x \in D \quad \text{and} \ u(x) = 0, \ x \in \partial D,$$

defines a system $\{G_\alpha^0\}$ of non-negative, bounded linear operators on $C(\overline{D})$ by $G_\alpha^0 u(x) = \int_D g_\alpha(x, y)\, u(y)\, dy$, $x \in \overline{D}$, where dy is the volume element of the Riemannian metric induced by the tensor $\{a^{ij}(x)\}$. G_α maps $C^H(\overline{D})$ into $C^2(\overline{D})$ and has the common range $\mathfrak{R}^0$ independent of the choice of $\alpha \ge 0$. Since $LG_\alpha^0 u \in C(\partial D)$ for $u \in C^H(\overline{D})$, we extend L to L' defined on $\mathfrak{R}^0$ by $L'u = \lim_{n\to\infty} Lu_n$ independently of the choice of $\{u_n\}$, where $u_n \to u$, $u_n = G_{\alpha_n}^0 v_n$, $v_n \in C^H(\overline{D})$.

We can prove $H_\alpha f - H_\beta f + (\alpha - \beta) G_\alpha^0 (H_\beta f) \equiv 0$ for $f \in C(\partial D)$, which implies $\overline{LH_\alpha}$, $\alpha > 0$ has the common domain $\tilde{\mathfrak{D}}$, and that

$$\overline{LH_\alpha} f - \overline{LH_\beta} f + (\alpha - \beta) L' G_\alpha^0 (H_\beta f) \equiv 0, \quad f \in \tilde{\mathfrak{D}}. \tag{7}$$

Let $\mathfrak{D}(\overline{L})$ be the space of all linear combinations $u = \sum_{k=1}^n u_k + \sum_{l=1}^m v_l$, $u_k \in \mathfrak{R}^0$ and $v_l = H_{\alpha_l} f_l$, $f_l \in \tilde{\mathfrak{D}}$. Then, by virtue of (7), we can define $\overline{L}u = \sum_{k=1}^n L' u_k + \sum_{l=1}^m \overline{LH_\alpha} f_l$ for $u \in \mathfrak{D}(\overline{L})$ independently of the representations of $u = \sum_{k=1}^n u_k + \sum_{l=1}^m v_l$. Then, we have

Theorem 2. *Assume that the condition of Theorem* 1 *holds for each* $\alpha > 0$, *and define* G_α *for* $u \in C(\overline{D})$ *and* $\alpha > 0$ *by*

$$G_\alpha u = G_\alpha^0 u - H_\alpha (\overline{LH_\alpha})^{-1} (\overline{L} G_\alpha^0 u).$$

Then, $\{G_\alpha\}$ *is a unique system of non-negative linear operators on*

$C(\bar{D})$ with norm $\|G_\alpha\| \leq \frac{1}{\alpha}$, satisfying

$$G_\alpha - G_\alpha + (\alpha-\beta)G_\alpha G_\beta = 0,$$
$$(\alpha - \bar{A})G_\alpha u = 0 \text{ and } \bar{L}G_\alpha u = 0 \text{ for } u \in C(\bar{D}),$$
$$\lim_{\alpha\to\infty} \alpha G_\alpha u = u \text{ for } u \in C(\bar{D}).$$

By virtue of Hille-Yosida theorem, Theorem 2 implies

Theorem 2′. *If the condition of Theorem 1 holds for each $\alpha > 0$, there is a unique semi-group $\{T_t\}$ of non-negative linear operators T_t on $C(\bar{D})$ with norm $\|T_t\| \leq 1$, strongly continuous in $t \geq 0$, with the generator $\mathfrak{G}$ which is a contraction of $\bar{A}$, and satisfying*

$$Lu(x) = 0,\ x \in \partial D, \text{ for } u \in C^2(\bar{D}) \frown \mathfrak{D}(\mathfrak{G}).$$

4. *A reduction to an integro-differential equation on the boundary.* Let $\widetilde{\widetilde{LH_\alpha}}$ be the restriction of LH_α on $C^3(\partial D)$ and let $\overline{\overline{LH_\alpha}}$ be its smallest closed extension. To solve (5) or (6) of Theorem 1, we represent $\widetilde{\widetilde{LH_\alpha}}$ in a more convenient form. To rewrite $\frac{\partial}{\partial n}H_\alpha$, we have

Lemma. *If there is a semi-group of non-negative linear operators $\{T_t\}$ on $C(\partial D)$ with norm at most one, strongly continuous in $t \geq 0$, with the generator $\widetilde{\mathfrak{G}}$ satisfying: $1, \xi_x^i(y)$, $1 \leq i \leq N-1$, $\sum_{i=1}^{N-1} (\xi_x^i(y))^2 \in \mathfrak{D}(\widetilde{\mathfrak{G}})$, for an $x \in \partial D$, then we have*

$$\widetilde{\mathfrak{G}}f(x) = \sum_{i,j=1}^{N-1} \alpha_1^{ij}(x) \frac{\partial^2 f}{\partial \xi^i \partial \xi^j}(x) + \sum_{i=1}^{N-1} \beta_1^i(x) \frac{\partial f}{\partial \xi^i}(x) + \gamma_1(x) f(x)$$
$$+ \int_{\partial D} \Big\{ f(y) - f(x) - \sum_{i=1}^{N-1} \frac{\partial f}{\partial \xi^i}(x) \xi^i(y) \Big\} \nu_{1,x}(dy), \qquad f \in C^3(\partial D) \frown \mathfrak{D}(\widetilde{\mathfrak{G}}),$$

where $\{\alpha_1^{ij}(x)\}$ is non-negative definite, $\gamma_1(x) \leq 0$ and $\nu_{1,x}(\cdot)$ is a measure on ∂D satisfying $\nu_{1,x}(\partial D - U_x) < +\infty$, $\sum_{i=1}^{N-1} (\xi_x^i(y))^2 \nu_{1,x}(dy) < +\infty$ for each neighbourhood U_x of x.

Since there is a unique semi-group on $C(\partial D)$ with the generator $\overline{\frac{\partial}{\partial n} H}$[2] satisfying the above conditions (by virtue of classical boundary value problem and Theorem 1), we represent $\widetilde{\widetilde{\frac{\partial}{\partial n} H}}$ by putting $\alpha_1^{ij} = \alpha_\alpha^{ij}$, $\beta_1^i = \beta_\alpha^i$, $\gamma_1 = \gamma_\alpha$ and $\nu_{1,x}(\cdot) = \nu_x^\alpha(\cdot)$. Moreover, write $\tilde{\beta}_\alpha^i(x) = \int_D \{H_\alpha[\xi_x^i]_{\partial D}(y) - \xi_x^i(y)\} \nu_x(dy)$, $\tilde{\gamma}_\alpha(x) = \int_D \{H_\alpha 1(y) - 1\} \nu_x(dy)$, $\bar{\alpha}_\alpha^{ij}(x) = \alpha^{ij}(x) + \alpha_\alpha^{ij}(x)$, $\bar{\beta}_\alpha^i(x) = \beta^i(x) + \beta_\alpha^i(x) + \tilde{\beta}_\alpha^i(x)$, $\bar{\gamma}_\alpha(x) = \gamma(x) + \alpha\delta(x) + \gamma_\alpha(x) + \tilde{\gamma}_\alpha(x)$, where $[\xi_x^i]_{\partial D}$ is the restriction of ξ_x^i on ∂D. Define a measure $\tilde{\nu}_x^\alpha(\cdot)$ on ∂D by

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2) This is a special case when $L = \frac{\partial}{\partial n}$.

$\int_{\partial D} f(y)\tilde{\nu}_x^\alpha(dy)=\int_D H_\alpha f(y)\nu_x(dy)$ and put $\bar{\nu}_x^\alpha(\cdot)=\nu(\cdot\frown\partial D)+\nu_x^\alpha(\cdot)+\tilde{\nu}_x^\alpha(\cdot)$.
Now, we have

Theorem 3. i) *For any* $f\in\mathfrak{D}(\widetilde{\widetilde{LH}}_\alpha)$ *we have*

$$\widetilde{\widetilde{LH}}_\alpha f(x)=\sum_{i,j=1}^{N-1}\bar{\alpha}_\alpha^{ij}(x)\frac{\partial^2 f}{\partial\xi^i\partial\xi^j}(x)+\sum_{i=1}^{N-1}\bar{\beta}_\alpha^i(x)\frac{\partial f}{\partial\xi^i}(x)+\bar{\gamma}_\alpha(x)f(x)$$
$$+\int_{\partial D}\Big\{f(y)-f(x)-\sum_{i=1}^{N-1}\frac{\partial f}{\partial\xi^i}(x)\xi_x^i(y)\Big\}\bar{\nu}_x^\alpha(dy),$$

where each coefficient and $\bar{\nu}_x^\alpha(\cdot)$ *satisfy the conditions similar to those in Lemma.* ii) *Fix an* $\alpha\geq 0$. *If there is a dense subspace* $\mathfrak{D}_\alpha$ *of* $C(\partial D)$ *such that the equation* $(\beta-\widetilde{\widetilde{LH}}_\alpha)g=f$ *or more generally* $(\beta-\overline{\overline{LH}}_\alpha)g=f$ *has a solution* $g\in\mathfrak{D}(\overline{\overline{LH}}_\alpha)$, *then the condition of Theorem* 1 *is satisfied.*

5. *Examples.* a) When the boundary condition is given by

$$Lu(x)=\gamma(x)u(x)+\delta(x)\lim_{y\to x}Au(y)+\mu(x)\frac{\partial u}{\partial n}(x)=0,$$

with sufficiently smooth γ, δ and μ, the condition of Theorem 1 is satisfied by virtue of a result of S. Ito [4].

b)[3] *When D is an open solid sphere in R^4 or an open disc in R^2, A is rotation invariant, and in* (4) $\alpha^{ij}(x)$, $\beta^i(x)$, $\mu(x)$ *are constants,* $\nu_x(\cdot)$, $\{\xi_x^i(y)\}$ *are rotation invariant, then the solution always exists.* In case $\gamma(x)$ and $\delta(x)$ are constants, we can construct the corresponding semi-group on $C(\partial D)$ with the generator $\mathfrak{G}_\alpha$ using Theorem 3 and G. Hunt [3, p. 279]. Next, we have a modified semi-group on $C(\partial D)$ with the generator $(\mathfrak{G}_\alpha+\gamma(x)+\alpha\delta(x))$ and get the result applying Theorem 2.

c) Take a set $\{(x_1,x_2,x_3)\in R^3,\ x_1^2+x_2^2<1,\ 0\leq x_3\leq 2\pi\}$ *and identify any point* $(x_1,x_2,0)$ *with* $(x_1,x_2,2\pi)$. *The set of such points D has the 2-dimensional torus as the boundary. Assume A and* $\alpha^{12}(x)=\alpha^{21}(x)$, $b^2(x)$, $\mu(x)$, *measure* $\nu_\alpha(\cdot)$ *and the local coordinates* $\{\xi_x^i(y)\}$ *are invariant under translations along x_3-axis and rotations around x_3-axis. If the coefficients are sufficiently smooth,* then the solution exists. Proof is similar to case b) using a result of K. Sato [5].

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3) Cf. Wentzell [7] and for a similar special case cf. T. Ueno [6].

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