152. The Space of Bounded Solutions and Removable Singularities of the Equation $\Delta u+au_x+bu_y+cu=0$ $(c\leqq 0)$

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1. Let D be a bounded domain in the complex z-plane. We consider a triple (a, b, c) where a and b are twice continuously differentiable functions and c is a non-positive, continuously differentiable function defined in a domain containing the closure of D.[1] We say that such a triple is *admissible*. Consider the partial differential equation of elliptic type:

(1) $$\Delta u+au_x+bu_y+cu=0,$$

where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, $u_x=\partial u/\partial x$ and $u_y=\partial u/\partial y$. Using notations of exterior differentials, (1) can be written as follows:

$$Lu=d^*du+du_\wedge\alpha+u\beta=0,$$

where $\alpha=-bdx+ady$ and $\beta=cdxdy$.

We denote by $B(a, b, c; D)$ the totality of bounded solutions of the equation (1) in D. Here a solution of (1) is always assumed to be twice continuously differentiable. Then $B(a, b, c; D)$ is a Banach space with the norm $\|u\|=\sup_D|u|$ (see [1]). Take another admissible triple $(\bar{a}, \bar{b}, \bar{c})$. In this note, we shall prove that $B(a, b, c; D)$ is isomorphic with $B(\bar{a}, \bar{b}, \bar{c}; D)$ as Banach spaces. In Nakai [5], this comparison problem was considered for triples $(0, 0, c)$ on a Riemann surface under some condition for c. Finally we shall characterize sets of removable singularities for bounded solutions of (1).[2]

2. Let $\{D_n\}_{n=1}^{\infty}$ be an exhaustion of D, i.e. D_n is a subdomain of D whose closure $\bar{D}_n$ is contained in D and whose boundary ∂D_n consists of a finite number of closed smooth Jordan curves and moreover $\{D_n\}_{n=1}^{\infty}$ satisfies

$$\bar{D}_n\subset D_{n+1} \quad\text{and}\quad D=\bigcup_{n=1}^{\infty}D_n.$$

Let $G_n(\zeta, z)$ be the Green function of (1) with respect to D_n with pole at ζ. It is well known that $G_n(\zeta, z)$ is the Green function of the adjoint equation of (1)

(1*) $$L^*u=d^*du-du_\wedge\alpha+(\beta-d\alpha)u=0$$

with respect to D_n with pole at z and, for each pair (ζ, z) in D, the sequence $\{G_n(\zeta, z)\}$ converges non-decreasingly to $G(\zeta, z)$ which is a

1) Functions considered in this note are all assumed to be real-valued.
2) The author extends his hearty thanks to Mr. Nakai for his kind suggestions.

solution of (1) in z and a solution of (1*) in ζ (see [3] and [4]). Moreover $G(\zeta, z)$ is bounded outside a neighbourhood of z as a function of ζ. We shall call $G(\zeta, z)$ the *Green function* with respect to D.

Let S be a closed disk with center z in D. Then we can prove

Lemma 1. *There exists a positive constant K for each point z in D such that*

$$\iint_{D_n - S} \left[\left(\frac{\partial G_n}{\partial \xi}(\zeta, z)\right)^2 + \left(\frac{\partial G_n}{\partial \eta}(\zeta, z)\right)^2\right] d\xi d\eta < K$$

for all n satisfying $D_n \supset S$, where $\zeta = \xi + i\eta$.

Proof. Fix a point z in D. Let u be a solution of (1*) in $D_n - \{z\}$ and vanishing on ∂D_n. For such a function u, we obtain

$$du_\wedge {}^*du = d\left(u^*du - \frac{1}{2}u^2\alpha\right) + u^2\left(\beta - \frac{1}{2}d\alpha\right). \tag{2}$$

Integrating (2) on $D_n - S$, we have

$$\iint_{D_n - S} du_\wedge {}^*du = \int_{\partial S}\left(u^*du - \frac{1}{2}u^2\alpha\right) + \iint_{D_n - S} u^2\left(\beta - \frac{1}{2}d\alpha\right), \tag{3}$$

since u vanishes on ∂D_n. Applying (3) to $G_n(\zeta, z)$, we get the assertion of Lemma 1 from the boundedness of a, b, $\partial a/\partial\xi$, $\partial b/\partial\eta$, and c in D and from the uniform boundedness of $G_n(\zeta, z)$, $\partial G_n(\zeta, z)/\partial\xi$ and $\partial G_n(\zeta, z)/\partial\eta$ on ∂S.

Lemma 2. (i) *If f is a bounded continuous function in D, then, for each point z in D,*

$$\lim_n \iint_{D_n} G_n(\zeta, z) f(\zeta) d\xi d\eta = \iint_D G(\zeta, z) f(\zeta) d\xi d\eta < \infty$$

and

$$\lim_n \iint_{D_n} \frac{\partial G_n}{\partial \xi}(\zeta, z) f(\zeta) d\xi d\eta = \iint_D \frac{\partial G}{\partial \xi}(\zeta, z) f(\zeta) d\xi d\eta < \infty.$$

(ii) *If a uniformly bounded sequence $\{f_n\}$ of continuous functions in D converges to a function f defined in D uniformly on every compact subset of D, then for each point z in D*

$$\lim_n \iint_{D_n} G_n(\zeta, z)(f_n(\zeta) - f(\zeta)) d\xi d\eta = 0$$

and

$$\lim_n \iint_{D_n} \frac{\partial G_n}{\partial \xi}(\zeta, z)(f_n(\zeta) - f(\zeta)) d\xi d\eta = 0.$$

Proof of (i). We prove the second identity since the first is trivial. Fix a point z in D. By Lemma 1 and Fatou's lemma, we can see easily

$$\iint_{D - S} \left[\left(\frac{\partial G}{\partial \xi}(\zeta, z)\right)^2 + \left(\frac{\partial G}{\partial \eta}(\zeta, z)\right)^2\right] d\xi d\eta \leqq K. \tag{4}$$

For a compact set A of D which contains S, the Schwarz inequality and Lemma 1 imply

$$(5)\quad \begin{aligned}\Big|\iint_{D_n-A}\frac{\partial G_n}{\partial\xi}(\zeta,z)f(\zeta)d\xi\,d\eta\Big|^2 &\leqq \iint_{D_n-A}\Big(\frac{\partial G_n}{\partial\xi}\Big)^2 d\xi d\eta\\ &\quad\times\iint_{D_n-A}|f|^2 d\xi d\eta\\ &\leqq K\cdot\sup_D|f|^2\cdot(\text{Area of }(D_n-A))\end{aligned}$$

for sufficiently large n. By the same argument as above, we get, using (4),

$$(6)\quad \Big|\iint_{D-A}\frac{\partial G}{\partial\xi}(\zeta,z)f(\zeta)d\xi d\eta\Big|^2\leqq K\cdot\sup_D|f|^2\cdot(\text{Area of }(D-A)).$$

On the other hand $\frac{\partial G_n}{\partial\xi}(\zeta,z)$ converges to $\frac{\partial G}{\partial\xi}(\zeta,z)$ uniformly on each compact set of D as n tends to infinity. Hence we get

$$(7)\quad \iint_A\Big(\frac{\partial G}{\partial\xi}-\frac{\partial G_n}{\partial\xi}\Big)f(\zeta)d\xi d\eta\to 0\quad (n\to\infty).$$

From (5), (6) and (7), we can conclude (i) of Lemma 2.

Proof of (ii). By our assumption there exists a constant M independent of n such that $|f_n(\zeta)|<M$ in D. If we apply (5) with $f=f_n-f$, we obtain

$$\Big|\iint_{D_n-A}\frac{\partial G_n}{\partial\xi}(\zeta,z)(f_n(\zeta)-f(\zeta))d\xi d\zeta\Big|^2\leqq 4KM^2\cdot(\text{Area of }(D_n-A)).$$

On A, the sequence $\partial G_n/\partial\xi\cdot(f_n-f)$ converges to 0 uniformly. Thus we get the second equality. The first identity is obvious. Therefore, we can conclude (ii) of Lemma 2.

3. **Theorem 1.** *For any two admissible triples* (a,b,c) *and* $(\bar a,\bar b,\bar c)$, *Banach spaces* $B(a,b,c;D)$ *and* $B(\bar a,\bar b,\bar c;D)$ *are isomorphic.*

Proof. Let $\bar G_n(\zeta,z)$ and $\bar G(\zeta,z)$ be Green functions with respect to D_n and D corresponding to the triple $(\bar a,\bar b,\bar c)$ respectively. For a bounded continuous function f in D, we define transformations Tf and tf as follows:

$$\begin{aligned}Tf(z)=f(z)+\frac{1}{2\pi}\iint_D&[(c(\zeta)-\bar c(\zeta))\bar G(\zeta,z)+\{(a(\zeta)-\bar a(\zeta))\\ &\times\bar G(\zeta,z)\}_\xi+\{(b(\zeta)-\bar b(\zeta))\bar G(\zeta,z)\}_\eta]f(\zeta)d\xi d\eta\end{aligned}$$

and

$$\begin{aligned}tf(z)=f(z)+\frac{1}{2\pi}\iint_D&[(\bar c(\zeta)-c(\zeta))G(\zeta,z)+\{(\bar a(\zeta)-a(\zeta))\\ &\times G(\zeta,z)\}_\xi+\{(\bar b(\zeta)-b(\zeta))G(\zeta,z)\}_\eta]f(\zeta)d\xi d\eta.\end{aligned}$$

By (i) of Lemma 2, we see that $Tf(z)<\infty$ and $tf(z)<\infty$ for each point z in D. We also define auxiliary transformations T_nf and t_nf of a bounded continuous function f defined in D_n as follows:

$$T_nf(z)=f(z)+\frac{1}{2\pi}\iint_{D_n}[(c(\zeta)-\bar{c}(\zeta))\bar{G}_n(\zeta, z) + \{(a(\zeta)-\bar{a}(\zeta))\bar{G}_n(\zeta, z)\}_\xi + \{(b(\zeta)-\bar{b}(\zeta))\bar{G}(\zeta, z)\}_\eta]f(\zeta)d\xi d\eta$$

and

$$t_nf(z)=f(z)+\frac{1}{2\pi}\iint_{D_n}[(\bar{c}(\zeta)-c(\zeta))G_n(\zeta, z) + \{(\bar{a}(\zeta)-a(\zeta))G_n(\zeta, z)\}_\xi + \{(\bar{b}(\zeta)-b(\zeta))G_n(\zeta, z)\}_\eta]f(\zeta)d\xi d\eta.$$

If h is continuous on $\bar{D}_n$ and is a solution of $Lu=0$ (or $\bar{L}u=d^*du+du_\wedge\bar{\alpha}+u\bar{\beta}=0$; $\bar{\alpha}=-\bar{b}dx+\bar{a}dy$, $\bar{\beta}=\bar{c}dxdy$) in D_n, then T_nh (or t_nh) is continuous on $\bar{D}_n$ and satisfies the equation $\bar{L}u=0$ (or $Lu=0$) in D_n and also $T_nh=h$ (or $t_nh=h$) on ∂D_n. Consequently we obtain

(8) $$\|T_nh\|_{D_n}=\|h\|_{D_n} \quad (\text{or } \|t_nh\|_{D_n}=\|h\|_{D_n}),$$
$$t_n(T_nh)=h \quad (\text{or } T_n(t_nh)=h).$$

On the other hand, if a uniformly bounded sequence $\{f_n\}$ of continuous function f_n in D converges to a function f defined in D uniformly on every compact subset of D, then for each point z in D

(9) $$Tf(z)=\lim_n T_nf_n(z) \quad (\text{or } tf(z)=\lim_n t_nf_n(z)).$$

In fact, setting

$$a_n(z)=|Tf(z)-T_nf(z)|,$$
$$b_n(z)=|T_nf(z)-f(z)-T_nf_n(z)+f_n(z)|,$$

and

$$c_n(z)=|f_n(z)-f(z)|,$$

we have

$$\lim_n a_n(z)=0$$

from (i) of Lemma 2 and

$$\lim_n b_n(z)=0$$

from (ii) of Lemma 2. Thus, using $\lim_n c_n(z)=0$ and

$$|Tf(z)-T_nf_n(z)|\leqq a_n(z)+b_n(z)+c_n(z),$$

we have (9).

Now take a function u in $B(a,b,c;D)$ (or $B(\bar{a},\bar{b},\bar{c};D)$). From (8), the sequence $\{T_nu\}$ (or $\{t_nu\}$) is bounded by $\|u\|$ in the absolute value and T_nu (or t_nu) is a solution of $\bar{L}u=0$ (or $Lu=0$). Hence by (9), T_nu (or t_nu) converges uniformly to Tu (or tu) on each compact subset of D which is a solution of $\bar{L}u=0$ (or $Lu=0$).

From (8) we have

(10) $$t_n(T_nu)=u \quad (\text{or } T_n(t_nu)=u).$$

If we apply (9) to (10) with $f_n=T_nu$, we see

$$t(Tu)=u \quad (\text{or } T(tu)=u),$$
$$\|tu\|\leqq\|u\| \quad (\text{or } \|Tu\|\leqq\|u\|).$$

This shows that T (or t) is a one-to-one mapping of $B(a,b,c;D)$ (or $B(\bar{a},\bar{b},\bar{c};D)$) onto $B(\bar{a},\bar{b},\bar{c};D)$ (or $B(a,b,c;D)$) and that $T=t^{-1}$. It is also obvious that both T and t are isometric. Thus Banach spaces $B(a,b,c;D)$ and $B(\bar{a},\bar{b},\bar{c};D)$ are isomorphic. This completes the proof of Theorem 1.

Assume that a part Γ of ∂D consists of a finite number of smooth closed Jordan curves. In this case, we denote by $B^{\Gamma}(a,b,c;D)$ the subspace of $B(a,b,c;D)$ consisting of every function in $B(a,b,c;D)$ which vanishes continuously on Γ. With an obvious modification of the proof of Theorem 1, we can prove the following

Theorem 1′. *Banach spaces $B^{\Gamma}(a,b,c;D)$ and $B^{\Gamma}(\bar{a},\bar{b},\bar{c};D)$ are isomorphic.*

4. A compact set E of D is said to be *(a,b,c)-removable* if, for any subdomain $\mathfrak{D}$ of D containing E, any bounded solution u of $Lu=0$ on a component $\mathfrak{D}_E$ of $\mathfrak{D}-E$ whose boundary contains the boundary of $\mathfrak{D}$ can be continued to a solution of $Lu=0$ on $\mathfrak{D}$. In this definition we may assume without loss of generality that the boundary $\partial\mathfrak{D}$ of $\mathfrak{D}$ consists of a finite number of smooth closed Jordan curves. As an application of our comparison theorem we prove

Theorem 2. *Let (a,b,c) be any admissible triple. Then a compact set of D is (a,b,c)-removable if and only if the logarithmic capacity of E equals zero.*

Proof. Let (a,b,c) and $(\bar{a},\bar{b},\bar{c})$ be any two admissible triples in D. Assume that E is (a,b,c)-removable. Let v be an arbitrary element in $B(\bar{a},\bar{b},\bar{c};\mathfrak{D}_E)$. We may assume without loss of generality that v is continuous on $\partial\mathfrak{D}\cup\mathfrak{D}_E$. Let v' be continuous on $\bar{\mathfrak{D}}$ and $v'=v$ on $\partial\mathfrak{D}$ and $\bar{L}v'=0$ in $\mathfrak{D}$. Putting $v''=v'-v$, we see that v'' is in $B^{\partial\mathfrak{D}}(\bar{a},\bar{b},\bar{c};\mathfrak{D}_E)$. On the other hand, by the maximum principle and by Theorem 1′, we have

$$B^{\partial\mathfrak{D}}(a,b,c;\mathfrak{D}_E)=B^{\partial\mathfrak{D}}(\bar{a},\bar{b},\bar{c};\mathfrak{D}_E)=\{0\}.$$

Hence $v''=0$ or $v'=v$ on $\mathfrak{D}_E$. Thus E is $(\bar{a},\bar{b},\bar{c})$-removable. By the same method, we easily see that if E is $(\bar{a},\bar{b},\bar{c})$-removable, then E is (a,b,c)-removable.

Taking $(\bar{a},\bar{b},\bar{c})=(0,0,0)$ and noticing that $(0,0,0)$-removable set is nothing but a set of logarithmic capacity zero, we can assure our Theorem.

The sufficiency of this theorem was proved by Inoue [2]. In the case of pairs $(0,0,c)$, this theorem was proved by Nakai [5] and Ozawa [6].

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