9. On Inner Automorphisms of Certain Finite Factors

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1. Employing the terminology of J. Dixmier [1], let us consider an abelian von Neumann algebra A with a faithful normal trace. If G is an ergodic group of automorphisms of A preserving the trace, then the crossed product $G\otimes A$ in the sense of T. Turumaru [5] coincides with the classical examples of finite factors due to F. J. Murray and J. von Neumann.

For an inner automorphism of $G\otimes A$ preserving the subalgebra A, I. M. Singer [4] proved that the inducing unitary operator $\Sigma_g g\otimes e_g$ satisfies certain properties; roughly speaking, up to a multiplication function, the character space of A splits into mutually disjoint clopen sets with the characteristic function e_g, on each of which the action of the automorphism coincides with the action of g.

2. Now, if A is a finite factor and G is an enumerable group of outer automorphisms[1] of A, then $G\otimes A$ is a finite factor.[2] The purpose of the present note is to show a factor analogue of Singer's theorem in the following

THEOREM. *If a unitary operator $\Sigma_g\otimes a_g$ induces an inner automorphism α of $G\otimes A$ which preserves the factor A, then all g-coefficients a_g vanish up to a certain g_0.*

Proof. If the unitary operator induces the action $x\to x^\alpha$, then

(1) $$(\Sigma_g g\otimes a_g)x=x^\alpha(\Sigma_g g\otimes a_g)$$

for all $x\in A$. (1) implies at once,

(2) $$a_g x=x^{\alpha g}a_g,$$

for all $x\in A$. In (2), if αg is known being outer as an action on A, then [2, Lemma 1] implies at once $a_g=0$. Hence, to prove the theorem, it is sufficient to show that g is outer on A for all $g\in G$ up to a certain g_0.

If not, then there is an another $g_1\in G$ for which αg_1 is inner too, or $\alpha g_1\equiv 1$ modulo the group I of all inner automorphisms of A. Hence, our hypothesis implies $\alpha g_0\equiv\alpha g_1$ mod. I, whence $g_0\equiv g_1$ mod. I. This is clearly impossible by the definition of the group G of outer automorphisms unless $g_0=g_1$. This proves the theorem.

3. By the above proof, the theorem can be restated as follows:

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1) A group G is called a group of outer automorphisms if each $g\in G$ is an outer automorphism unless $g=1$.

2) A proof of the statement is contained in [2, Theorem 1].

COROLLARY. *An A-preserving inner automorphisms of the crossed product $G\otimes A$ of a finite factor A by an enumerable group G of outer automorphisms coincides with a member of the group G up to an inner automorphism of A.*

It is to be noticed that the proof of the theorem implies also that an automorphism α of $G\otimes A$ preserving A is outer on $G\otimes A$ if αg is outer on A for all $g\in G$. This remark gives an another proof of [3, §6, Corollary].

References

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