3. Uniform Extension of Uniformly Continuous Functions

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In this note, a space is uniform and a function is, unless otherwise specified, real valued and uniformly continuous.

Katětov proved [3, Theorem 3] that, if A is an arbitrary uniform subspace of a space S, then any bounded function on A can be uniformly extended to S. In this note, we are going to find conditions under which the same kind of extension holds for not necessarily bounded functions. In other words, when we say that a space S has *a property E* if any function on an arbitrary uniform subspace of S can be uniformly extended to S, then we shall see in the following some conditions of S in order to have the property E. A space is said to be *uc* if every real valued continuous function on the space is uniformly continuous. Some characterisations for a space to be *uc* are known [1]. When S is normal and *uc*, then S has the property E, this is a trivial sufficient condition. Another sufficient condition is well known [2, Theorem 4.12], which is however not necessary even in a metric space. Theorem 2 gives a necessary and sufficient condition in a pseudo-metric space, and it also induces a necessary and sufficient condition of a space to have a restricted property E.

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The following theorem is a corollary to the Katětov's theorem [3, Theorem 3], which however gives a sufficient condition for the property E which does not induce the local fineness [2].

Theorem 1. *Let $\{f^\alpha\}$ be a uniformly equicontinuous family of functions f^α on a uniform subspace A of a space S into closed intervals $[a^\alpha, b^\alpha]$, $0<b^\alpha-a^\alpha=c^\alpha<c<\infty$, then there is a uniformly equicontinuous family of uniform extensions of f^α to S.*

Proof. Let S' be the union of disjoint copies S^α of S for all α, then the family of unions V'_β of disjoint copies V^α_β of all entourages V_β in S generates a uniform structure in S', and f defined by $f^\alpha-c^\alpha$ is uniformly continuous on $\smile A^\alpha$, A^α copies of A, to $[0, c]$. By the Katětov's theorem, there is a uniform extension g of f to S'. $g^\alpha+c^\alpha$ is desired extension of f^α.

We can prove, in an elementary way similar to the well-known proof of the Urysohn's extension theorem in normal spaces, that the range of the extension of f^α in this theorem is also $[a^\alpha, b^\alpha]$.

A family $\{X^\alpha\}$ of subsets is *uniformly discrete* if the sets $V(X^\alpha)$ are pairwise disjoint for some entourage V, and a space is *finitely chainable* [1] if, for any entourage V, there are finitely many points $p_1,\cdots,p_m$ and a natural number n such that $\{V^n(p_i);\ 1\leq i\leq m\}$ covers the space. We know [1] that every function on a space is bounded if and only if the space is finitely chainable. Then we have

Corollary 1. *Let A be a uniform subspace of a space S which is decomposed into a uniformly discrete family of uniform subspaces S^α, and f a function on A such that the diameters of $f(A\frown S^\alpha)$'s are less than a positive number for all α, then f can be uniformly extended to S.*

Corollary 2. *Let a space S be decomposed into a uniformly discrete family of finitely chainable uniform subspaces S^α such that for any entourage V $S^\alpha\times S^\alpha\subset V$ for all but finite number of α, then S has the property E.*

The following example shows that the condition that $\{c^\alpha\}$ is bounded in Theorem 1 cannot be dropped. Let S be the union of the closed intervals $[2n, 2n+1]$ on the real line, A the set of the natural numbers $\{2n+1/3, 2n+2/3;\ n=1,2,\cdots\}$, and f^n the mapping on A with the values $f^n(2n+1/3)=(2n-1)^2$, $f^n(2n+2/3)=(2n)^2$, and 0 for other points, then $\{f^n\}$ is uniformly equicontinuous on A without desired extension.

Applying Corollary 2 to the space $\smile[n, n+1/n]$, we shall see that a space with the property E is not necessarily *uc* (cf. [1]) or locally fine (cf. [2, Theorem 5.5]).

Definition 1. Let f be a function defined on a uniform subspace A of a space S. *A modulus of uniform continuity* (or simply *modulus*) of f is a sequence of entourages $V_1, V_2,\cdots$ in S, $V_n^2\subset V_{n-1}$, $V_n^{-1}=V_n$, such that $(x,y)\in V_m$ implies $|f(x)-f(y)|<1/n$ on A for any n and some m. We say that f is *uniformly modulus-preservingly extensible* if, for any modulus of f, there is a uniform extension of f to S which has a modulus consisting of members of the modulus of f. A space has *a property E'* if every function on any uniform subspace of the space is uniformly modulus-preservingly extensible.

Suppose that a uniform structure of S is generated by a family M of pseudo-metrics, and f is a bounded function on a uniform subspace of S. Then, since the uniform continuity of f is determined by its modulus, f is uniformly continuous on the uniform subspace of S considered as a pseudo-metric space defined by some pseudo-metric in M, and, by the Katětov's theorem, f can be uniformly extended to S; namely, if we restrict ourselves to bounded functions, every uniform space has the property E'. In a similar consideration, we can easily see that f in Corollary 1 is uniformly modulus-preservingly extensible with a modulus each member of which is contained in some entourage, and S in Corollary 2 has the property E' with respect to moduli contained in some entourage. In these circumstances, it will be natural to investigate first the conditions of a space to have the property E', for not necessarily bounded functions.

As we have seen above, *in order that a space with a uniform structure generated by a family M of pseudo-metrics has the property E', it is necessary and sufficient that every pseudo-metric space defined by a pseudo-metric in M has the property E.* Moreover, any space has the property E if and only if its completion does. Therefore the following Theorem 2 is, in this case, essential.

Definition 2. A family of subsets of a pseudo-metric space is said to be *e-discrete*, e a positive number, if the distance of any two members of the family is not less than e. $V_{1/n}$ is the entourage of the space consisting of pairs of points whose distances are less than $1/n$, and $V^{\infty}_{1/n}=\bigcup_m V^m_{1/n}$.

Theorem 2. *A pseudo-metric complete space has the property E if and only if, for any natural number n, there is a compact subset K such that for any open subset G containing K there is a natural number m satisfying $V_{1/n}(p)\supset V^{\infty}_{1/m}(p)$ for every point $p\notin G$.*

Lemma 2. *If a pseudo-metric space S has the property E, then for any natural numbers m, n, and $1/m$-discrete infinite subset D there is a natural number m_0 such that $V_{1/n}(p)\supset V^{\infty}_{1/m_0}(p)$ in S for all but finite points p in D.*

Proof. Suppose, to the contrary, that there are natural numbers m', n' and a $1/m'$-discrete subset D such that, for any natural number m, $V_{1/n'}(p)\not\supset V^{\infty}_{1/m}(p)$ for infinitely many points p in D. Let n be a natural number satisfying $nn'\geq 2m'$, then, by the induction, we can take countably many distinct points p_i, $i>2nn'$, in D such that $V_{1/n'}(p_i)\not\supset V^{\infty}_{1/i}(p_i)$ for each i. Let us take x_i in S such that $V_{1/n'}(p_i)\not\ni x_i\in V^{\infty}_{1/i}(p_i)$, then there are $x^0_i=p_i, x^1_i,\cdots, x^h_i=x_i$ with $d(x^j_i, x^{j+1}_i)<1/i$, d the pseudo-metric of our space. The last point $x^{k_i}_i$ included in $V_{1/nn'}(p_i)$, which is disjoint from every $V_{1/nn'}(p_j)$, $i\neq j$, does not belong to any $V_{1/2nn'}(p_h)$. Put

$$A=\{S-\bigcup_{i>2nn'} V_{1/2nn'}(p_i)\}\cup\{p_i;\, i>2nn'\},$$

$$f(x)=\begin{cases}0 & \text{for } x\in A-\{p_i;\, i>2nn'\},\\ k_i & \text{for } x=p_i,\end{cases}$$

then f is uniformly continuous on A. If there is a uniform extension g of f to S, then there is a natural number m such that $d(x,y)<1/m$ implies $|g(x)-g(y)|<1$. For i greater than m and $2nn'$, we have $k_i=|f(p_i)-f(x^{k_i}_i)|=|g(p_i)-g(x^{k_i}_i)|<k_i$, contradiction. Consequently, f cannot have any uniform extension to S.

Proof of Theorem 2. Suppose that the space has the property E. Let n be an arbitrary natural number and put

$K_i=\{p;\ V_{1/n}(p)\not\supset V^{\infty}_{1/i}(p)\},\ i=1,2,\cdots,$

$K=\frown \bar{K}_i.$

By Lemma 2, there is no uniformly discrete sequence of points in K, i.e. K is compact. Let G be any open set containing K and let all K_i be not contained in G, then there is an infinite set $\{x_i \in K_i\}$ of points which are not included in G. By Lemma 2, $\{x_i\}$ cannot contain any uniformly discrete subsequence, i.e. $\{x_i\}$ is precompact, and it has an accumulation point not included in G. Since $K_i \supset K_{i+1}$, the accumulation point is included in K, impossible. Consequently, we have $K_m \subset G$ for some m, and $V_{1/n}(p) \supset V^{\infty}_{1/m}(p)$ for every $p \notin G$. Conversely, suppose the space S satisfies the conditions of our assertion. Let A be a closed uniform subspace, and f a function defined on A. There is a natural number n such that $d(x,y)<2/n$ implies $|f(x)-f(y)|<1$ on A. There are a compact set K and a natural number m such that $V_{1/n}(p) \supset V^{\infty}_{1/m}(p)$ for any point $p \notin V_{1/n}(K)=G$.

$$\{S-\smile_{p \notin G} V^{\infty}_{1/m}(p),\ V^{\infty}_{1/m}(p);\ p\notin G\}$$

is a 1/m-discrete decomposition of the space, f is bounded on

$$A\frown\{S-\smile_{p\notin G} V^{\infty}_{1/m}(p)\},$$

and $|f(x)-f(y)|<1$ for any x and y in $A \frown V^{\infty}_{1/m}(p)$, $p\notin G$. Therefore, by Corollary 1 to Theorem 1, f can be uniformly extended to the whole space.

Examining the above proof of sufficiency, we can easily see that this kind of condition is also sufficient in general spaces: *a uniform space has the property E if for any entourage V there is a precompact subset K such that for any open set G containing K there is an entourage W satisfying $V(p)\supset W^{\infty}(p)$ for every point $p\notin G$.* A space satisfying this condition is not always locally fine, because, as we see in the last example after Corollary 2, even a complete metric space with the condition in Theorem 2 is not necessarily locally fine.

References

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