24. On Distribution Solution of Partial Differential Equations of Evolution. II

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We shall continue the study of the properties of the classes $\mathfrak{E}^s_t\mathfrak{D}'$ in section 3 and prove the main theorem 6 in section 4.

3. THEOREM 3. *Let* $(G_n)_0$ *be a subdomain of* G_n *and* $a\leqq a_0<b_0\leqq b$ *and let* $\tilde{T}\in\mathfrak{E}^s_t\mathfrak{D}'(G_{n+1})$ $(+\infty\geqq s>-\infty)$. *Then the restriction* $(\tilde{T})_0$ *of* $\tilde{T}$ *on* $(G_{n+1})_0(=(G_n)_0\times(a_0, b_0))$ *belongs to* $\mathfrak{E}^s_t\mathfrak{D}'[(G_{n+1})_0]$.

PROOF. The proof follows immediately from the definitions of the classes $\mathfrak{E}^s_t\mathfrak{D}'$, so we omit the proof of Theorem 3.

THEOREM 4. *Let* $(G_n)_0$ *be a domain in* R^n *such that* $\overline{(G_n)_0}\subseteq G_n$ *and* $\overline{(G_n)_0}$ *is compact. Also let* $-\infty\leqq a<a_0<b_0<b\leqq+\infty$. *If* $\tilde{T}\in\mathfrak{D}'(G_{n+1})$, *then there is an integer* s *such that the restriction* $(\tilde{T})_0$ *of* $\tilde{T}$ *on* $(G_{n+1})_0(=(G_n)_0\times(a_0, b_0))$ *belongs to* $\mathfrak{E}^s_t\mathfrak{D}'[(G_{n+1})_0]$.

PROOF. By the local structure theorem of distributions,[1] we can find a complex-valued function $F_0\in C^0[(G_{n+1})_0]$ such that $(\tilde{T})_0=D^{s'}_t D^{\alpha}_x F_0$ s' an integer$\geqq 0$. By Lemma 2, F_0 regarded as a distribution belongs to $\mathfrak{E}^0_t\mathfrak{D}'[(G_{n+1})_0]$. Hence by (2.6) in Theorem 2 and by Theorem 1, we have $(\tilde{T})_0\in\mathfrak{E}^s_t\mathfrak{D}'[(G_{n+1})_0]$ $s=-s'$. Q.E.D.

THEOREM 5. *Let* $\tilde{T}\in\mathfrak{D}'(G_{n+1})$. *Assume that each point* (x_0, t_0) *of* G_{n+1} *has a neighbourhood* $(G_{n+1})_0$ *of the form* $(G_n)_0\times(a_0, b_0)$ *where* $-\infty\leqq a\leqq a_0<b_0\leqq b\leqq+\infty$ *and* $(G_n)_0$ *is a subdomain of* G_n *such that the restriction* $(\tilde{T})_0$ *of* $\tilde{T}$ *on* $(G_{n+1})_0$ *belongs to* $\mathfrak{E}^s_t\mathfrak{D}'[(G_{n+1})_0]$ *where* s $(-\infty<s\leqq+\infty)$ *is the same for all points* $(x_0, t_0)\in G_{n+1}$. *Then* $\tilde{T}\in\mathfrak{E}^s_t\mathfrak{D}'(G_{n+1})$.

PROOF. For $+\infty\geqq s\geqq 0$, the proof of Theorem 5 is immediate if a suitable partition of the unity[2] on G_{n+1}, the univalency of the mapping M^{-1} and the compactness of the carriers of the test functions φ for the distribution $\tilde{T}$ are used. Hence we omit the proof for the case.

For $-\infty<s<0$, we proceed as follows. For $\tilde{T}\in\mathfrak{D}'(G_{n+1})$, there exists always a distribution $\tilde{T}_3\in\mathfrak{D}'(G_{n+1})$ such that $\tilde{T}=D^{-s}_t\tilde{T}_3$.[3] Then

1) Cf. L. Schwartz [2], p. 83.
2) Cf. L. Schwartz [2], p. 23.
3) Cf. L. Schwartz [2], p. 55. The same remark as in 7) applies here also.

the restriction $(\tilde{T}_3)_0$ of $\tilde{T}_3$ on $(G_{n+1})_0$ belongs to $\mathfrak{E}_t^0\mathfrak{D}'[(G_{n+1})_0]$ by the premiss of Theorem 5 and by Theorem 1. Hence $\tilde{T}_3 \in \mathfrak{E}_t^0\mathfrak{D}'(G_{n+1})$ since for $s=0$ Theorem 5 is already proved. Therefore by Theorem 1, we get $\tilde{T}=D_t^{-s}\tilde{T}_3 \in \mathfrak{E}_t^s\mathfrak{D}'(G_{n+1})$. Q.E.D.

4. Let $a_{i,j,\alpha}(x,t)\in C^\infty(G_{n+1})$, $\tilde{B}_i=\mathfrak{M}[(B_i)_t]\in\mathfrak{E}_t^0\mathfrak{D}'(G_{n+1})$ and $\tilde{U}_i=\mathfrak{M}[(U_i)_t]\in\mathfrak{E}_t^1\mathfrak{D}'(G_{n+1})$ for $i,j=1,\cdots,n$ and $|\alpha|\leqq l$[4] where l is anon-negative integer. Then by Lemmas 3, 4 and 6,

$$D_t\tilde{U}_i=\sum_{|\alpha|\leqq l}\sum_{j=1}^n a_{i,j,\alpha}(x,t)D_x^\alpha\tilde{U}_j+\tilde{B}_i \quad i=1,\cdots,n \tag{4.1}$$

on G_{n+1}, if and only if

$$d(U_i)_t/dt=\sum_{|\alpha|\leqq l}\sum_{j=1}^n a_{i,j,\alpha}(x,t)D_x^\alpha(U_j)_t+(B_i)_t \quad i=1,\cdots,n$$

on (a,b).

Also let $a_{i,j,\alpha}(x,t)\in C^\infty(G_{n+1})$, $\tilde{B}_i\in\mathfrak{D}'(G_{n+1})$ and $\tilde{U}_i\in\mathfrak{E}_t^s\mathfrak{D}'(G_{n+1})$ for $i,j=1,\cdots,n$ and $|\alpha|\leqq l$[4] where s is an integer or $+\infty(-\infty<s\leqq+\infty)$ and l is a non-negative integer. Then if (4.1) is satisfied on G_{n+1}, then $\tilde{B}_i\in\mathfrak{E}_t^{s-1}\mathfrak{D}'(G_{n+1})$ $i=1,\cdots,n$ by Theorems 1, 2 and Lemma 9.

We prove a converse of the later statement in Theorem 6. The answer for the problem stated in the introduction is given by the case $s=1$ of Theorem 6.

THEOREM 6. *Let $a_{i,j,\alpha}(x,t)\in C^\infty(G_{n+1})$, $\tilde{B}_i\in\mathfrak{E}_t^{s-1}\mathfrak{D}'(G_{n+1})$ and $\tilde{U}_i\in\mathfrak{D}'(G_{n+1})$ for $i,j=1,\cdots,n$ and $|\alpha|\leqq l$[4] where s is an integer or $+\infty$ $(-\infty<s\leqq+\infty)$, l is a non-negative integer and G_{n+1} is a domain in (x,t)-space of the form $G_n\times(a,b)$. If $\tilde{U}_i$ satisfy on G_{n+1} the system of partial differential equations of evolution*

$$D_t\tilde{U}_i=\sum_{|\alpha|\leqq l}\sum_{j=1}^n a_{i,j,\alpha}(x,t)D_x^\alpha\tilde{U}_j+\tilde{B}_i \quad i=1,\cdots,n, \tag{4.1}$$

then $\tilde{U}_i\in\mathfrak{E}_t^s\mathfrak{D}'(G_{n+1})$ $i=1,\cdots,n$.

PROOF. Let (x_0,t_0) be any point of G_{n+1}. We take a neighbourhood $(G_{n+1})_0$ of the form $(G_n)_0\times(a_0,b_0)$ of (x_0,t_0) where $-\infty\leqq a<a_0<b_0<b\leqq+\infty$, $\overline{(G_n)_0}\subseteq G_n$ and $\overline{(G_n)_0}$ is compact. We denote the restrictions of $\tilde{U}_i$ and of $\tilde{B}_i$ on $(G_{n+1})_0$ by $(\tilde{U}_i)_0$ and $(\tilde{B}_i)_0$ respectively. By Theorem 5, for the proof of Theorem 6 it is sufficient to prove that $(\tilde{U}_i)_0\in\mathfrak{E}_t^s\mathfrak{D}'[(G_{n+1})_0]$ for every point $(x_0,t_0)\in G_{n+1}$.

Assume the contrary, that is, assume that at least one of $(\tilde{U}_i)_0$ does not belong to $\mathfrak{E}_t^s\mathfrak{D}'[(G_{n+1})_0]$ for a point $(x_0,t_0)\in G_{n+1}$. Then by Theorem 4, there is the greatest s_0 of integers s such that all $(\tilde{U}_i)_0$ belong to $\mathfrak{E}_t^{s'}\mathfrak{D}'[(G_{n+1})_0]$, and $s>s_0$. Therefore on $(G_{n+1})_0$, the right

4) α_k $(k=1,\cdots,n)$ non-negative integers $\alpha=(\alpha_1,\cdots,\alpha_n)$
$|\alpha|=\sum_{k=1}^n\alpha_k$ $D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$.

sides of (4.1) belong to $\mathfrak{E}_t^{s_0}\mathfrak{D}'[(G_{n+1})_0]$, since $(\tilde{B}_i)_0 \in \mathfrak{E}_t^{s-1}\mathfrak{D}'[(G_{n+1})_0] \subseteq \mathfrak{E}_t^{s_0}\mathfrak{D}'[(G_{n+1})_0]$ by Theorem 3 and Lemma 9 and also other terms in the right sides of (4.1) belong to $\mathfrak{E}_t^{s_0}\mathfrak{D}'[(G_{n+1})_0]$ on $(G_{n+1})_0$ by Theorem 2. Hence the left sides of (4.1) on $(G_{n+1})_0$, $D_t(\tilde{U}_i)_0 \in \mathfrak{E}_t^{s_0}\mathfrak{D}'[(G_{n+1})_0]$ $i=1,\cdots,n$ so that by Theorem 1, $(\tilde{U}_i)_0 \in \mathfrak{E}_t^{s_0+1}\mathfrak{D}'[(G_{n+1})_0]$ $i=1,\cdots,n$. But this contradicts the definition of s_0. Q.E.D.

References

[1] L. Schwartz: Les équations d'évolution liées au produit de composition, Ann. Inst. Fourier, II, 19–49 (1950).

[2] L. Schwartz: Théorie des Distributions, I, Hermann, Paris (1950).

[3] L. Schwartz: Théorie des Distributions, II, Hermann, Paris (1950).

[4] C. Chevalley: Theory of Distributions, Lecture notes at Columbia University (1950–1951).