23. On Distribution Solution of Partial Differential Equations of Evolution. I

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1. **Introduction.** When the solutions $u_i(x, t)$ of a system of linear partial differential equations of evolution

(1.1) $D_t u_i = \sum_{|\alpha| \leq l} \sum_{j=1}^{n} a_{i,j,\alpha}(x, t) D_x^\alpha u_j + b_i(x, t) \quad i=1,\cdots,n$

($\alpha_k (k=1,\cdots,n)$ non-negative integers $\alpha=(\alpha_1,\cdots,\alpha_n)$ $|\alpha|=\sum_{k=1}^{n}\alpha_k$ $x=(x_1,\cdots,x_n)$ $D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$ D_{x_i}, D_t: the operators of partial differentiation with respect to x_i and to t, and l: a non-negative integer) are discussed, $u_i(x, t)$ are sometimes[1] considered as continuously differentiable functions of t whose values are distributions in (x)-space in the sense of L. Schwartz. But coordinate transformations mixing the space coordinates x_i and the time coordinate t are important for some problems. For such problems, solutions in different coordinate systems are compared most naturally by considering them as distributions in (x, t)-space in the sense of L. Schwartz. Not only for such reasons but also by itself, it is of some interest to ask: when can a distribution solution u_i $(i=1,\cdots,n)$ in (x, t)-space of a system of equations of evolution (1.1) where $a_{i,j,\alpha}(x, t)$ are infinitely differentiable functions of (x, t) and $b_i(x, t)$ are distributions in (x, t)-space be considered as a set of continuously differentiable functions of t whose values are distributions in (x)-space?

The main theorem 6 in section 4 of this note shows that this is the case, if and only if in (1.1) $b_i(x, t)$ are distributions in (x, t)-space which can be considered as continuous functions of t whose values are distributions in (x)-space.[2] Theorem 6 contains also more precise results. If $a_{i,j,\alpha}(x, t)$ are infinitely differentiable, all distributions $u_i(x, t)$ in (x, t)-space constituting a solution of (1.1) belong to a class $\mathfrak{E}_t^{s+1}\mathfrak{D}'$ generally by one step more regular with respect to t than a class $\mathfrak{E}_t^s\mathfrak{D}'$ $(+\infty \geqq s > -\infty)$ (but $\mathfrak{E}_t^{s+1}\mathfrak{D}'=\mathfrak{E}_t^s\mathfrak{D}'$, if $s=+\infty$) to which all distributions $b_i(x, t)$ in (x, t)-space in the right sides of (1.1) belong. Cf. Definitions 3 and 5. Also every distribution in (x, t)-space belongs locally to a class $\mathfrak{E}_t^s\mathfrak{D}'$ $(+\infty \geqq s > -\infty)$ by Theorem 4.

As preparations to section 4, we shall classify distributions in (x, t)-space according to their regularity with respect to t and prove related theorems in sections 2 and 3.

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1) For example, L. Schwartz [1].

2) In section 4, we shall give a precise formulation of the above statements.

2.[3] We begin with some notations.

$\mathfrak{D}(G)$: the linear space of infinitely differentiable complex-valued functions with compact carriers in a domain G. Its topology is that given in L. Schwartz [2].

$\mathfrak{D}'(G)$: the linear space of distributions on G, that is, the dual space of $\mathfrak{D}(G)$. Its topology is that given in L. Schwartz [2].

$C^m(G)$: the class of $m(\infty \geqq m \geqq 0)$ times continuously differentiable complex-valued functions on G.

In this note, G_n is always a domain in x-space $(=R^n)$ and (a, b) is always an open interval in t-space such that $-\infty \leqq a < b \leqq +\infty$. We write also the domain $G_n \times (a, b)$ in (x, t)-space as G_{n+1}.

DEFINITION 1. *We denote by* $\mathfrak{E}_t^m[(a, b), G_n]$ $(+\infty \geqq m \geqq 0)$ *the linear space of* m *times continuously differentiable*[4] *functions of* t *on* (a, b) *with values in* $\mathfrak{D}'(G_n)$.[5]

We denote the derivative[4] $\in \mathfrak{E}_t^0[(a, b), G_n]$ of a function $T_t \in \mathfrak{E}_t^1[(a, b), G_n]$ of t by dT_t/dt and the distribution derivatives of a distribution $\tilde{T} \in \mathfrak{D}'(G_{n+1})$ by $D_{x_i}\tilde{T}, D_t\tilde{T}$. Also we denote the distribution derivatives of a distribution $T \in \mathfrak{D}'(G_n)$ by $D_{x_i}T$. Also if $\alpha_k (k=1, \cdots, n)$ are non-negative integers, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $|\alpha| = \sum_{k=1}^n \alpha_k$.

If $T_t \in \mathfrak{E}_t^0[(a, b), G_n]$, then T_t defines a distribution $\tilde{T} \in \mathfrak{D}'(G_{n+1})$ in (x, t)-space by

$$\tilde{T}(\varphi) = \int_a^b T_t[\varphi(x, t)]dt \tag{2.1}$$

where $\varphi(x, t) \in \mathfrak{D}(G_{n+1})$.

DEFINITION 2. *We denote by* $\mathfrak{M}$ *the linear mapping of* $\mathfrak{E}_t^0[(a, b), G_n]$ *into* $\mathfrak{D}'(G_{n+1})$ *defined by* (2.1).

DEFINITION 3. *We denote by* $\mathfrak{E}_t^m\mathfrak{D}'(G_{n+1})$ $(+\infty \geqq m \geqq 0)$ *the image of* $\mathfrak{E}_t^m[(a, b), G_n]$ *by* $\mathfrak{M}$. $\mathfrak{E}_t^m\mathfrak{D}'(G_{n+1})$ $(+\infty \geqq m \geqq 0)$ *is a linear subspace of* $\mathfrak{D}'(G_{n+1})$.

In section 2, we abbreviate $\mathfrak{E}_t^m[(a, b), G_n]$, $\mathfrak{E}_t^m\mathfrak{D}'(G_{n+1})$ $(+\infty \geqq m \geqq 0)$ as $\mathfrak{E}_t^m$, $\mathfrak{E}_t^m\mathfrak{D}'$ respectively since we are concerned with only one domain G_n in (x)-space and with only one interval (a, b) in t-space and with only one domain $G_{n+1} = G_n \times (a, b)$ in (x, t)-space in this section.

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3) In section 2, we shall state some definitions and lemmas. The proofs of these lemmas and of the consistency of these definitions follow easily from the fundamental properties of distributions as given in L. Schwartz [2, 3] so that we shall almost always omit the proofs.

4) We define the derivative dT_t/dt of a function T_t of t with values in $\mathfrak{D}'(G_n)$ by $(T_{t+\Delta t} - T_t)/\Delta t \to dT_t/dt (\Delta t \to 0)$ in the strong or the weak topology of $\mathfrak{D}'(G_n)$. Here it becomes the same thing if we consider the continuity and the differentiability of the function T_t in the strong or the weak topology of $\mathfrak{D}'(G_n)$. Cf. L. Schwartz [2], p. 75.

5) We denote the class of infinitely differentiable functions of t on (a, b) with values in $\mathfrak{D}'(G_n)$ by $\mathfrak{E}_t^{+\infty}[(a, b), G_n]$ since we shall define later classes $\mathfrak{E}_t^s\mathfrak{D}'(G_{n+1})$ also for negative integers s.

LEMMA 1. *The mapping* $\mathfrak{M}$ *is one to one. Hence if* $\tilde{T}\in\mathfrak{E}_t^m\mathfrak{D}'$ $(+\infty\geqq m\geqq 0)$, *then* $\mathfrak{M}^{-1}(\tilde{T})\in\mathfrak{E}_t^m$.

By Lemma 1, we have

$$(2.2)\qquad \bigcap_{+\infty>m\geqq 0}\mathfrak{E}_t^m\mathfrak{D}'=\mathfrak{E}_t^{+\infty}\mathfrak{D}'.$$

LEMMA 2. *Let* F *be a complex-valued function* $\in C^0(G_{n+1})$. *Then* F *regarded as distribution* $\in\mathfrak{D}'(G_{n+1})$ *belongs to* $\mathfrak{E}_t^0\mathfrak{D}'$.

LEMMA 3. *If* $T_t\in\mathfrak{E}_t^0$ *and so* $\tilde{T}=\mathfrak{M}(T_t)\in\mathfrak{E}_t^0\mathfrak{D}'$, *then* $D_x^\alpha T_t\in\mathfrak{E}_t^0$, $D_x^\alpha\tilde{T}\in\mathfrak{E}_t^0\mathfrak{D}'$ *and* $D_x^\alpha\tilde{T}=\mathfrak{M}(D_x^\alpha T_t)$.

LEMMA 4. *If* $T_t\in\mathfrak{E}_t^1$ *and so* $\tilde{T}=\mathfrak{M}(T_t)\in\mathfrak{E}_t^1\mathfrak{D}'$, *then* $D_t\tilde{T}=\mathfrak{M}(dT_t/dt)$.

LEMMA 5. *If* $T_t\in\mathfrak{E}_t^1$, *then* $d(D_x^\alpha T_t)/dt$ *exists and* $d(D_x^\alpha T_t)/dt=D_x^\alpha(dT_t/dt)$. *Hence by Lemma 3,* $D_x^\alpha T_t\in\mathfrak{E}_t^1$.

LEMMA 6. *If* $T_t\in\mathfrak{E}_t^0$ *and so* $\tilde{T}=\mathfrak{M}(T_t)\in\mathfrak{E}_t^0\mathfrak{D}'$ *and if* $a(x,t)\in C^\infty(G_{n+1})$, *then* $a(x,t)T_t\in\mathfrak{E}_t^0$, $a(x,t)\tilde{T}\in\mathfrak{E}_t^0\mathfrak{D}'$ *and* $a(x,t)\tilde{T}=\mathfrak{M}[a(x,t)T_t]$.

We denote the Riemann integral[6] on an interval $[a',b']$ $(a<a'<b'<b)$ of a function $T_t\in\mathfrak{E}_t^0$ by

$$\int_{a'}^{b'}T_t\,dt.$$

Also we define

$$\int_{a'}^{b'}T_t\,dt=-\int_{b'}^{a'}T_t\,dt \text{ if } a'>b' \text{ and } \int_{a'}^{a'}T_t\,dt=0.$$

Then if $a<c,t<b$,

$$\int_c^t T_t\,dt\in\mathfrak{E}_t^1 \text{ and } \frac{d}{dt}\int_c^t T_t\,dt=T_t.$$

DEFINITION 4. *If* $T_t\in\mathfrak{E}_t^0$ *and so* $\tilde{T}=\mathfrak{M}(T_t)\in\mathfrak{E}_t^0\mathfrak{D}'$ *and if* $a<c<b$, *we define the operation* $I_{c,t}$ *on* $\tilde{T}$ *by*

$$I_{c,t}(\tilde{T})=\mathfrak{M}\Big(\int_c^t T_t\,dt\Big).$$

Then by Lemma 4, we have

LEMMA 7. *If* $\tilde{T}\in\mathfrak{E}_t^0\mathfrak{D}'$, *then* $I_{c,t}(\tilde{T})\in\mathfrak{E}_t^1\mathfrak{D}'$ *and* $D_tI_{c,t}(\tilde{T})=\tilde{T}$.

LEMMA 8. *If* $\tilde{T}\in\mathfrak{D}'(G_{n+1})$ *and* $D_t\tilde{T}=0$, *then* $\tilde{T}$ *is of the form*

$$\tilde{T}(\varphi)=\int_a^b T(\varphi(x,t))\,dt$$

where $\varphi(x,t)\in\mathfrak{D}(G_{n+1})$ *and* T *is a distribution* $\in\mathfrak{D}'(G_n)$.[7] *Hence* $\tilde{T}\in\mathfrak{E}_t^{+\infty}\mathfrak{D}'$.

6) We define the Riemann integral on an interval $[a',b']$ $(a<a'<b'<b)$ of a function $T_t\in\mathfrak{E}_t^0$ by the limit of the Riemann sum when the maximum length of the subintervals belonging to the subdivision of $[a',b']$ corresponding to the Riemann sum tends to zero. The limit exists in the strong topology of $\mathfrak{D}'(G_n)$. Cf. C. Chevalley [4], p. 36.

7) Lemma 8 is proved in L. Schwartz [2], p. 113 for the case $G_n\times(a,b)=R^n\times R$. But the generalization to the present case is immediate.

Now we define the classes $\mathfrak{E}_t^m\mathfrak{D}'(G_{n+1})$ of distributions for negative integers m as follows.

DEFINITION 5. *We denote by* $\mathfrak{E}_t^{-m}\mathfrak{D}'(G_{n+1})$ *(m positive integer) the linear space of all distributions* $\tilde{T}\in\mathfrak{D}'(G_{n+1})$ *of the form* $\tilde{T}=D_t^m\tilde{T}_1$ *where* $\tilde{T}_1\in\mathfrak{E}_t^0\mathfrak{D}'(G_{n+1})$.

In section 2, we abbreviate $\mathfrak{E}_t^s\mathfrak{D}'(G_{n+1})$ as $\mathfrak{E}_t^s\mathfrak{D}'$ for $0>s>-\infty$ as well as for $+\infty\geqq s\geqq 0$.

LEMMA 9. *If* $+\infty\geqq s'\geqq s>-\infty$, *then*

$$\mathfrak{E}_t^s\mathfrak{D}'\supseteq\mathfrak{E}_t^{s'}\mathfrak{D}'. \tag{2.3}$$

PROOF. For $+\infty\geqq s'\geqq s\geqq 0$, (2.3) is obvious. Hence it is sufficient to prove (2.3) for the case $0\geqq s'\geqq s>-\infty$. If $\tilde{T}\in\mathfrak{E}_t^{s'}\mathfrak{D}'$ and $0\geqq s'\geqq s>-\infty$, then we have a distribution $\tilde{T}_1\in\mathfrak{E}_t^0\mathfrak{D}'$ such that $\tilde{T}=D_t^{-s'}\tilde{T}_1$. Then $\tilde{T}=D_t^{-s'}[D_t^{s'-s}I_{c,t}^{s'-s}(\tilde{T}_1)]=D_t^{-s}[I_{c,t}^{s'-s}(\tilde{T}_1)]$ and $I_{c,t}^{s'-s}(\tilde{T}_1)\in\mathfrak{E}_t^{s'-s}\mathfrak{D}'\subseteq\mathfrak{E}_t^0\mathfrak{D}'$ by Lemma 7 and (2.3) for the case $+\infty\geqq s'\geqq s\geqq 0$. Q.E.D.

THEOREM 1. *Let* $\tilde{T}\in\mathfrak{D}'(G_{n+1})$. *Then* $\tilde{T}\in\mathfrak{E}_t^s\mathfrak{D}'(+\infty\geqq s>-\infty)$, *if and only if* $D_t\tilde{T}\in\mathfrak{E}_t^{s-1}\mathfrak{D}'$.

PROOF. If $\tilde{T}\in\mathfrak{E}_t^s\mathfrak{D}'$, then it follows immediately from the definitions of the classes $\mathfrak{E}_t^s\mathfrak{D}'$ and Lemma 4 that $D_t\tilde{T}\in\mathfrak{E}_t^{s-1}\mathfrak{D}'$. Hence we shall prove only the converse. Let $D_t\tilde{T}\in\mathfrak{E}_t^{s-1}\mathfrak{D}'$. If $s-1<0$, there is a distribution $\tilde{T}_2\in\mathfrak{E}_t^0\mathfrak{D}'$ such that $D_t\tilde{T}=D_t^{1-s}\tilde{T}_2$. Now we define a distribution $I_t(D_t\tilde{T})\in\mathfrak{D}'(G_{n+1})$ for the general case $-\infty<s\leqq+\infty$ as follows:

$$I_t(D_t\tilde{T})=\begin{cases}I_{c,t}(D_t\tilde{T}) & \text{if } s-1\geqq 0\\ D_t^{-s}\tilde{T}_2 & \text{if } s-1<0.\end{cases}$$

Then by Lemma 7 and Definition 5, we have always

$$I_t(D_t\tilde{T})\in\mathfrak{E}_t^s\mathfrak{D}' \tag{2.4}$$

$$D_t[I_t(D_t\tilde{T})]=D_t\tilde{T}. \tag{2.5}$$

From (2.5) we have $D_t[I_t(D_t\tilde{T})-\tilde{T}]=0$ so that by Lemmas 8 and 9, we get $I_t(D_t\tilde{T})-\tilde{T}\in\mathfrak{E}_t^{+\infty}\mathfrak{D}'\subseteq\mathfrak{E}_t^s\mathfrak{D}'$. From this and (2.4) we have $\tilde{T}\in\mathfrak{E}_t^s\mathfrak{D}'$. Q.E.D.

THEOREM 2. *If* $\tilde{T}\in\mathfrak{E}_t^s\mathfrak{D}'(+\infty\geqq s>-\infty)$ *and* $a(x,t)\in C^\infty(G_{n+1})$, *then*

$$D_x^\alpha\tilde{T}\in\mathfrak{E}_t^s\mathfrak{D}' \tag{2.6}$$

$$a(x,t)\tilde{T}\in\mathfrak{E}_t^s\mathfrak{D}'. \tag{2.7}$$

PROOF. The proof of (2.6) follows immediately from the definitions of the classes $\mathfrak{E}_t^s\mathfrak{D}'$ and Lemmas 3 and 5. Hence we prove only (2.7) by the induction on s. For $s=0$, (2.7) is already established by Lemma 6.

We begin with the case $+\infty>s\geqq 0$. Let (2.7) be already established for an integer $s\geqq 0$ and assume that $\tilde{T}\in\mathfrak{E}_t^{s+1}\mathfrak{D}'$. Then by the assumption of the induction, $D_t[a(x,t)\tilde{T}]=(\partial a(x,t)/\partial t)\tilde{T}+a(x,t)D_t\tilde{T}$ $\in\mathfrak{E}_t^s\mathfrak{D}'$, since $\tilde{T}, D_t\tilde{T}\in\mathfrak{E}_t^s\mathfrak{D}'$ by Lemma 9 and Theorem 1. Therefore by Theorem 1, we have $a(x,t)\tilde{T}\in\mathfrak{E}_t^{s+1}\mathfrak{D}'$. Thus the induction for the case $+\infty>s\geqq 0$ is completed. (2.7) for the case $s=+\infty$ follows from (2.7) for the case $+\infty>s\geqq 0$ and from (2.2).

Now we proceed to the case $0\geqq s>-\infty$. Let (2.7) be already established for an integer $s\leqq 0$ and assume that $\tilde{T}\in\mathfrak{E}_t^{s-1}\mathfrak{D}'$. Then there is a distribution $\tilde{T}_1\in\mathfrak{E}_t^0\mathfrak{D}'$ such that $\tilde{T}=D_t^{1-s}\tilde{T}_1$. Hence by Theorem 1 and Lemma 9, $a(x,t)\tilde{T}=D_t[a(x,t)D_t^{-s}\tilde{T}_1]-(\partial a(x,t)/\partial t)D_t^{-s}\tilde{T}_1\in\mathfrak{E}_t^{s-1}\mathfrak{D}'$ since $a(x,t)D_t^{-s}\tilde{T}_1$, $(\partial a(x,t)/\partial t)D_t^{-s}\tilde{T}_1\in\mathfrak{E}_t^s\mathfrak{D}'$ by the assumption of the induction. Thus the induction for the case $0\geqq s>-\infty$ is completed. Q.E.D.

References

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