19. Note on a Semigroup Having No Proper Subsemigroup

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In the previous paper [1] we determined the structure of $\mathfrak{S}$-semigroup and added that a finite semigroup of order >2 which contains no proper subsemigroup is a cyclic group of prime order. Further we noticed, without proof, that this holds even if the condition "finiteness" is excluded.

In the present note we shall prove the theorem without using the result of $\mathfrak{S}$-semigroup.

Theorem. *A semigroup of order*[1] >2 *which has no proper subsemigroup is a cyclic group of prime order.*

Let S be a semigroup of order >2 which has no proper subsemigroup, and let a and b be arbitrary distinct elements of S. Then we see that S is generated by a and b. First we must prove that S contains at least a non-idempotent element. For this purpose we may show that an idempotent semigroup M of order >2 generated by the two distinct elements a and b has at least one proper subsemigroup. Now, F denotes the free idempotent semigroup generated by a and b. M is given as a suitable[2] factor semigroup of F. Fortunately it is easily proved[3] that F is a semigroup of order 6 which consists of

$$a,\quad b,\quad ab,\quad ba,\quad aba,\quad bab.$$

As is easily seen, F has a proper subsemigroup, say $\{b, ab, ba, aba, bab\}$. Let us consider every decomposition[4] of F which raises a factor semigroup M.

If a or b alone composes a coset, the problem is clear, that is, if $\{a\}$ is a coset, all the other cosets form a proper subsemigroup of the factor semigroup M, because $\{b, ab, ba, aba, bab\}$ is a subsemigroup of F. If a and b belong to different cosets containing at least two elements, then we may examine only the decomposition such that

$$(1)\qquad a\sim aba,\quad b\sim bab,\quad \text{and}\quad a\not\sim b,$$

because the other factor semigroups of F would be of order at most 2. In detail,

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1) By "a semigroup of order >2" we mean "an infinite or finite semigroup which contains at least 3 elements".

2) We require a condition that a and b do not belong to the same coset.

3) See [2].

4) A classification of elements which gives a factor semigroup is called a decomposition of a semigroup. Each decomposition corresponds to each congruence relation.

$a \sim ab$ implies $a \sim ab \sim aba, \quad ba \sim bab$,

$a \sim ba$ implies $a \sim ba \sim aba, \quad ab \sim bab$,

$a \sim bab$ implies $b \sim ab \sim ba \sim aba \sim bab$,

$b \sim ab$ implies $b \sim ab \sim bab, \quad ba \sim aba$,

$b \sim ba$ implies $b \sim ba \sim bab, \quad ab \sim aba$,

$b \sim aba$ implies $b \sim ab \sim ba \sim aba \sim bab$.

Since these decompositions, except the third and the sixth, are mutually disjoint, the decompositions besides (1) give the factor semigroups of order>3.

In the case of the decomposition (1), we see $ab \not\sim ba$, $a \not\sim ab$, $a \not\sim ba$, $b \not\sim ab$, $b \not\sim ba$, since $ab \sim ba$ would imply $a \sim b$, and one of the other relations would imply that the order of M is less than 3. Hence M is given as the following table:

$A=\{a, aba\}$, $B=\{b, bab\}$, $C=\{ab\}$, $D=\{ba\}$,

	A	B	C	D
A	A	C	C	A
B	D	B	B	D
C	A	C	C	A
D	D	B	B	D

where $\{A, C\}$ is a proper subsemigroup. Thus it has been proved that if a semigroup M of order>2 is idempotent, M has a proper subsemigroup. Therefore, S contains a non-idempotent element if S of order>2 has no proper subsemigroup.

Now let a be a non-idempotent element of S. Considering the subsemigroup generated by a, it must coincide with S.

$$S=\{a^n \mid n=1, 2, \cdots\}.$$

This is finite, because the infinite power semigroup contains a proper subsemigroup, say, $\{a^{2n} \mid n=1, 2, \cdots\}$. It is well known that a finite power semigroup S contains a group G as the ideal. Accordingly, in this case we must have

$$S = G,$$

i.e. S is a finite group. It is an elementary problem to prove that a finite group having no proper subsemigroup is a cyclic group of prime order. Thus the theorem has been completely proved.

By the way, we should like to add the following remark:

It is not true that a quasigroup Q having no proper subset, which is closed with respect to the operation of Q, is a cyclic group of prime order.

For example, consider a quasigroup Q of order 4 defined as

	a	b	c	d
a	c	a	b	d
b	a	d	c	b
c	b	c	d	a
d	d	b	a	c .

This quasigroup is not a group and it is generated by any single element, in fact

$$\begin{aligned} &a^2=c, \quad ca=b, \quad c^2=d; \\ &b^2=d, \quad d^2=c, \quad cd=a; \\ &c^2=d, \quad cd=a, \quad ac=b; \\ &d^2=c, \quad cd=a, \quad ac=b. \end{aligned}$$

From this it follows that Q has no proper subset closed with respect to the operation. Furthermore, Q contains no idempotent element.

References

[1] T. Tamura and M. Sasaki: Finite semigroups in which Lagrange's theorem holds, Journal of Gakugei, Tokushima University, **5**, 33-38 (1959).
[2] D. McLean: Idempotent semigroups, Amer. Math. Monthly, **61**, 110-113 (1954).