18. On (m, n)-distributive Division Rings

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Following Beaumont [1], an additive abelian group R, closed with respect to multiplication, is called an *(m, n)-distributive ring* if the (m, n)-distributive law holds in R:

(1) $$(\textstyle\sum_{i=1}^{m} a_i)(\sum_{j=1}^{n} b_j)=\sum_{i,j=1}^{m,n} a_i b_j \quad \textit{for all } a_i, b_j \in R.$$

In other words, an (m, n)-distributive ring is a system arising from the definition of a (not necessarily associative) ring by replacing two distributive laws by the above (m, n)-distributive law. The structure of (m, n)-distributive rings was studied by Beaumont [1], Hsiang [2] and Saitô [3].

An (m, n)-distributive ring D is called an *(m, n)-distributive division ring* if D has at least two elements and $D-\{0\}$ forms a multiplicative group. In this note, we study the connection between (m, n)-distributive division rings and (ordinary) division rings.

We consider exclusively (m, n)-distributive rings for $m, n \geqq 2$. So, in this paper, *we assume that m and n are always integers $\geqq 2$.*

Theorem 1. *If an (m, n)-distributive division ring D contains at least three elements, then D is a division ring.*

Proof. Recall, in an (m, n)-distributive ring, we have

(2) $$(a+b)c=ac+bc-0c, \qquad c(a+b)=ca+cb-c0$$

([1], p. 877). By assumption, for any $x \in D^*=D-\{0\}$, we can take an element $y \in D^*$ such that $y \neq x$. We set $z=yx^{-1}$. Then $z-1 \in D^*$. Now, by the equality (2) and the associativity of multiplication in D^*, we have

$$\begin{aligned} yx^{-1}+(0x)x^{-1}-0x^{-1} &= (y+0x)x^{-1}=(zx+0x)x^{-1} \\ &= (((z-1)+1)x+0x)x^{-1} \\ &= ((z-1)x+1x-0x+0x)x^{-1} \\ &= ((z-1)x+x)x^{-1} \\ &= (z-1)xx^{-1}+xx^{-1}-0x^{-1} \\ &= (z-1)+1-0x^{-1}=yx^{-1}-0x^{-1}. \end{aligned}$$

Hence $(0x)x^{-1}=0$. By the closedness of multiplication in D^*, we have $0x=0$. Dually $x0=0$. Moreover, since

$$0=0z=0((z-1)+1)=0(z-1)+01-00=0-00,$$

we have $00=0$. Thus

$$0x=x0=0 \quad \text{for all } x \in D.$$

Hence, by (2), we have distributive laws:

$$(a+b)c=ac+bc, \qquad c(a+b)=ca+cb.$$

This completes the proof.

Theorem 2. *If both m and n are even, then every (m, n)-distributive division ring is a division ring.*

Proof. Let D be an (m, n)-distributive division ring which is not a division ring. Then, by Theorem 1, D has just two elements, which are clearly 0 and 1. Since $1+1\neq 1$, we have $1+1=0$. Now recall that, in an (m, n)-distributive ring R, if $\varphi(a)\equiv 0$, $\psi(a)\equiv 0$, $00=0$ where $\varphi(a)=a0-00$, $\psi(a)=0a-00$, then R is a ring (Theorem 1 of [3] or First Fundamental Theorem of [2]). Hence, in D, at least one of the relations

$$\varphi(1)=1,\quad \psi(1)=1,\quad 00=1$$

holds, since $\varphi(0)=\psi(0)=0$. Therefore, if both m and n were even, at least one of the relations

$$(n-1)\varphi(1)=1,\quad (m-1)\psi(1)=1,\quad (mn-1)00=1$$

would hold, which contradicts the fact that all the relations

$$(n-1)\varphi(a)\equiv 0,\quad (m-1)\psi(a)\equiv 0,\quad (mn-1)00=0 \tag{3}$$

hold in (m, n)-distributive ring (Lemma 1 of [3] or First Fundamental Theorem of [2]). This completes the proof.

Theorem 2 is a generalization of Theorem 2 of [4].

Theorem 3. (a) *If m is even and n odd, there is one and only one (m, n)-distributive division ring which is not a division ring. It is represented by the table*

(4)

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	1	1

.

(b) *If m is odd and n even, there is one and only one (m, n)-distributive division ring which is not a division ring. It is represented by the table*

(5)

+	0	1
0	0	1
1	1	0

·	0	1
0	0	1
1	0	1

.

(c) *If both m and n are odd, there are seven and only seven (m, n)-distributive division rings which are not division rings. Two of them are represented by the tables* (4) *and* (5) *and the remaining five have common addition table which is same as that of* (4) *and have the multiplication tables*

(6)

·	0	1
0	0	1
1	1	1

(7)

·	0	1
0	1	0
1	0	1

(8)

$\cdot$	0	1
0	1	1
1	1	1

(9)

$\cdot$	0	1
0	1	0
1	1	1

(10)

$\cdot$	0	1
0	1	1
1	0	1

.

Proof. Let D be an (m, n)-distributive division ring which is not a division ring. By the proof of Theorem 2, D has just two elements, 0 and 1, and it has the addition table of this theorem. If m is even and n odd, then, by (3), we have $00=0$ and $\psi(1)=0$. Hence $\varphi(1)=1$, for otherwise we would have $\varphi(a)\equiv 0$, $\psi(a)\equiv 0$ and $00=0$, and so D would be a division ring. Thus we obtain the table (4). Dually, if m is odd and n even, we obtain the table (5). Now we consider the case when both m and n are odd. $\varphi(a)$ is the identity mapping or zero mapping according as $\varphi(1)=1$ or $\varphi(1)=0$ respectively. Similarly, $\psi(a)$ is the identity mapping or zero mapping. Hence there are seven possibilities of choosing $\varphi(a)$, $\psi(a)$ and 00, since $\varphi(a)=0$, $\psi(a)=0$, $00=0$ must be excluded. Thus we obtain the tables (4)–(10). Conversely, the above systems are surely (m, n)-distributive division rings in each case, can be verified.

Remark. It can also be verified that all the systems of Theorem 3 except (9) and (10) are associative.

References

[1] R. A. Beaumont: Generalized rings, Proc. Amer. Math. Soc., **9**, 876–881 (1958).

[2] W. Hsiang: On the distributive law, Proc. Amer. Math. Soc., **11**, 348–355 (1960).

[3] T. Saitô: Note on generalized rings, to appear in Bull. Tokyo Gakugei Univ., **12** (1961).

[4] T. Saitô: Note on the distributive laws (supplement), unpublished to appear in Amer. Math. Monthly.