112. *Strongly p-Parabolic Systems*

By Yôjirô HASEGAWA
Kyoto University

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1. *Introduction.* As we know, I. G. Petrowsky [1] defined the p-parabolic single equation ($p \geqq 2$, integer) as follows:

$$\left[\left(\frac{\partial}{\partial t}\right)^m + \sum_{((k))} a^{(k_0,k)}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n}\right]u + \sum_{(k)} b^{(k_0,k)}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n} u = f(t, x_1,\cdots, x_n), \tag{1.1}$$

1) $k_0 < m$, $k=(k_1,\cdots, k_n)$; $pk_0+k_1+\cdots+k_n \leqq pm$, and where $\sum_{((k))}$ and $\sum_{(k)}$ mean the summations of the terms; $pk_0+k_1+\cdots+k_n = pm$, and the terms; $pk_0+k_1+\cdots+k_n < pm$, respectively.

2) The roots $\lambda_i(\xi)$ of the characteristic equation

$$\lambda^m + \sum_{((k))} a^{(k_0,k)}\lambda^{k_0}(i\xi)^k = 0$$

satisfy

(1.2) real part $\lambda_i(\xi) \leqq -\delta|\xi|^p$, for some positive constant δ,

where $\xi=(\xi_1, \xi_2,\cdots, \xi_n)$, $(i\xi)^k=(i\xi_1)^{k_1}\cdots(i\xi_n)^{k_n}$, and $|\xi| = \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}$.[1]

Petrowsky [1] proved that this forward Cauchy problem for this equation is well posed. On the other hand, he showed that, if there exist $\xi^0 \neq 0$ (real) and $\lambda_i(\xi)$ such that

(1.3) real part $\lambda_i(\xi^0) > 0$,

then the forward Cauchy problem for (1.1) is never well posed.[2]

Is the condition (1.2) necessary? This is not true. However, we want to show that the condition (1.2) is necessary, when we restrict ourselves to the strongly well posed equations (whose definition is given by definition 1.1).

Definition 1.1. *We say that* (1.1) *is strongly well posed* (*as a p-evolution equation*), *if* (1.1) *is well posed for any choice of the lower part*: what we call the lower part is

$$l\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1},\cdots, \frac{\partial}{\partial x_n}\right)u = \sum_{(k)} b^{(k_0,k)}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n} u.$$

Our purpose is to prove the following.

Theorem 1.1. *In order that the p-evolution equation* (1.1) ($p \geqq 2$,

1) It is clear that such a positive integer p satisfying the condition (1.2) must be even.

2) He proved this fact in the case where the coefficients depend only on t for the p-evolution systems.

integer) be strongly well posed for the future, it is necessary and sufficient that we have (1.2) *for some positive constant* δ.

The sufficiency was proved by Petrowsky [1], therefore we need only to show its necessity. Before proving our theorem (1.1), we remark the following fact: A necessary and sufficient condition in order that the forward Cauchy problem for the p-evolution equation (1.1) be well posed is the following: Denote the roots of the equation

$$\lambda^m+\sum_{((k))} a^{(k_0,k)}\lambda^{k_0}(i\xi)^k+\sum_{(k)} b^{(k_0,k)}\lambda^{k_0}(i\xi)^k=0$$

by $\lambda_i^*(\xi)$, then the real parts of the $\lambda_i^*(\xi)$, $i=1,2,\cdots,m$, are bounded from above (see L. Gårding [2]).

2. *Proof of Theorem* 1.1. Consider the p-evolution equation consisting only of the principal part of (1.1),

$$p\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right)u=\left[\left(\frac{\partial}{\partial t}\right)^m+\sum_{((k))} a^{(k_0,k)}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n}\right]u,$$

then the characteristic equation corresponding to the above equation is

$$p(\lambda;i\xi)=\lambda^m+\sum_{((k))} a^{(k_0,k)}\lambda^{k_0}(i\xi)^k=0.$$

Our aim is to prove the following fact. Assume one of $\lambda_i(\xi)$, say $\lambda_1(\xi)$, become pure imaginary for some $\xi^0=\eta^0|\xi|$, $|\eta^0|=1$, then we can choose a suitable polynomial $b(i\xi)$ of degree $\leqq mp-1$, in such a way that the equation

(2.1) $$P(\lambda;i\xi)=p(\lambda;i\xi)+b(i\xi)=0$$

has at least one root, say $\lambda_1^*(\xi)$, such that

(2.2) real part $\lambda_1^*(\eta^0|\xi|)\to+\infty$, when $|\xi|\to+\infty$.

Put $\lambda=\mu|\xi|^p$, $\xi/|\xi|=\eta$, then, (2.1) is equivalent to

$$P(\mu;i\eta)=p(\mu;i\eta)+b(i\eta)|\xi|^{s-mp}=0,$$

where $s=$degree of $b(i\xi)$.

We take now,

(2.3) $$b(i\xi)=c(i\xi_l)^s$$

where l is an index such that $\eta_l^0\neq 0$, s is an integer such that

$$(m-\nu_1)p<s<mp,$$

(where ν_1 is the multiplicity of the root $\lambda_1(\xi^0)$ of $p(\lambda;i\xi^0)=0$), and c is a suitable constant (complex in general). Or it is equivalent to

(2.3′) $$b(i\xi)=c(i\eta_l)^s|\xi|^s.$$

In fact, put

$$p(\mu)=p(\mu;i\eta^0)=(\mu-\mu_1^0)^{\nu_1}\cdots(\mu-\mu_k^0)^{\nu_k}$$

where $\mu_j^0=\mu_j(\eta^0)$, $j=1,2,\cdots,k$, are roots of $p(\mu)=0$, and μ_1^0 is the root corresponding to the $\lambda_1(\eta^0|\xi|)$.

Then

$$P(\mu;|\xi|)=p(\mu)+c(i\eta_l^0)^s|\xi|^{s-mp}=(\mu-\mu_1^0)^{\nu_1}Q(\mu)+c(i\eta_l^0)^s|\xi|^{s-mp},$$

where $Q(\mu)=(\mu-\mu_2^0)^{\nu_2}\cdots(\mu-\mu_k^0)^{\nu_k}$, and $Q(\mu_1^0)\neq 0$.

This implies that $P(\mu;|\xi|)=0$ has the roots of the form;

$$(2.4)\qquad \mu^*=\mu_1^0+\left(-\frac{c(i\eta_l^0)^s}{Q(\mu_1^0)}\right)^{1/\nu_1}|\xi|^{\frac{s-mp}{\nu_1}}+o(|\xi|^{\frac{s-mp}{\nu_1}}).$$

Now, we can choose c in such a way that one of the quantity $(-c(i\eta_l^0)^s/Q(\mu_1^0))^{1/\nu_1}$ is real positive.[3)]

Hence,

$$(2.5)\quad \text{real part } \mu_1^*(\eta^0)=\text{real part } [(-c(i\eta_l^0)^s/Q(\mu_1^0))^{1/\nu_1} \times|\xi|^{\frac{s-mp}{\nu_1}}+o(|\xi|^{\frac{s-mp}{\nu_1}})]\geqq\delta'|\xi|^{\frac{s-mp}{\nu_1}},$$

where $\delta'>0$, and $\mu_1^*(\eta^0)$ is a root of $P(\mu; i\eta^0, |\xi|)=0$, or

$(2.5')$ real part $\lambda_1^*(\eta^0|\xi|)\to+\infty$. q.e.d.

This means that if a p-evolution (1.1) is strongly well posed, then the condition (1.2) is necessary.

3. *The case of a system of p-evolution equations.* Now we consider a system of p-evolution equations:

$$(3.1)\qquad \begin{aligned}\left(\frac{\partial}{\partial t}\right)^{m_i}u_i=&\sum_{j=1}^N\sum_{|k|=(m_j-k_0)p}a_{ij}^{(k_0,k)}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n}u_j\\&+\sum_{j=1}^N\sum_{|k|<(m_j-k_0)p}b_{ij}^{(k_0,k)}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n}u_j\\&+f_i(t,x_1,\cdots,x_n)\qquad(i=1,2,\cdots,N),\end{aligned}$$

where $|k|=\sum_{i=1}^n k_i$, $0\leqq k_0<m_j$, $k=(k_1,k_2,\cdots,k_n)$, and $a_{ij}^{(k_0,k)}$, $b_{ij}^{(k_0,k)}$ are constants.

Definition 3.1. We say that the system (3.1) is strongly well posed (as a p-evolution), if (3.1) is well posed for any choice of the lower part, where the lower part means those ones

$$\sum_{|k|<(m_j-k_0)p}b_{ij}^{(k_0,k)}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n}u_j.$$

Denote by $p_{ij}(\lambda; i\xi)$ the polynomial corresponding to the differential operators of the principal part, which has the form:

$$p_{ij}(\lambda; i\xi)=\delta_{ij}\lambda^{m_i}-\sum_{|k|=(m_j-k_0)p}a_{ij}^{(k_0,k)}\lambda^{k_0}(i\xi_1)^{k_1}\cdots(i\xi_n)^{k_n}.$$

Then the characteristic equation of the system (3.1) is

$$(3.2)\qquad \det(p_{ij}(\lambda; i\xi))=0.$$

Theorem 3.1. *In order that the system of p-evolution equations* (3.1) ($p\geqq2$, *integer*) *be strongly well posed for the future, it is necessary and sufficient that we have the following condition.*

For all the roots $\lambda_i(\xi)$ *of* (3.2), *there exists a positive constant* δ, *such that*

(3.3) real part $\lambda_i(\xi)\leqq-\delta|\xi|^p$ for all $\xi\,(\neq0)$ real.

4. *Proof of Theorem* 3.1. As well as the proof of the case of the single equation, put $\lambda=\mu|\xi|^p$, $\xi_i=\eta_i|\xi|$, then (3.2) is equivalent to

3) We give attention to the choice of c: in the case $\nu_1\gtrsim3$, c is arbitrarily chosen except zero, this implies that we can choose c as real. However, in the case where $\nu_1=1$, or $=2$, we cannot always take c as real. For example, $p\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial t}$, $p=2$.

(4.1) $$\det(p_{ij}(\mu; i\eta))=0.$$

We have $N_1 \left(=\sum_{j=1}^{N} m_j\right)$ roots $\mu_j(\eta)$ of the above equation.

Now we assume that one of $\mu_j(\eta)$, say $\mu_1(\eta)$, become pure imaginary for $\eta=\eta^0$, $|\eta^0|=1$.

Then

$$\det(p_{ij}(\mu; i\eta^0))=(\mu-\mu_1(\eta^0))^{\nu_1}\cdots(\mu-\mu_k(\eta^0))^{\nu_k},$$

where $\nu_i\,(\geqq 1)$ is the multiplicity of the root $\mu_i(\eta^0)$, and $\sum_{i=1}^{k}\nu_i=N_1$.

Put

(4.2) $$Q(\mu)=Q(\mu; i\eta^0)=(\mu-\mu_2(\eta^0))^{\nu_2}\cdots(\mu-\mu_k(\eta^0))^{\nu_k},$$

this never vanishes near η^0, (i.e. $Q(\mu)\neq 0$ at $\mu=\mu_1(\eta^0)$).

On the other hand we denote by $p^*_{ij}(\mu; i\eta^0)$ the cofactor of $p_{ij}(\mu; i\eta^0)$, and put $p^*_{ij}(\mu; i\eta^0)$;

(4.3) $$p^*_{ij}(\mu; i\eta^0)=(\mu-\mu_1(\eta^0))^{\nu_{ij}}q^*_{ij}(\mu; i\eta^0),$$

where $\nu_{ij}\geqq 0$, and then $q^*_{ij}(\mu; i\eta^0)\neq 0$ at $\mu=\mu_1(\eta^0)$.

Put

(4.4) $$\nu=\min_{i,j}\nu_{ij}.$$

Now we see that $\nu\leqq\nu_1-1$. In fact, the matrix $(p_{ij}(\mu; i\eta^0))$ is similar to the matrix

$$\begin{bmatrix} e_1(\mu) & & 0 \\ & \ddots & \\ 0 & & e_N(\mu) \end{bmatrix},$$

where $e_i(\mu)$; $i=1,2,\cdots,N$, are the elementary divisors of the matrix $(p_{ij}(\mu; i\eta^0))$.

Here we see that the elementary divisor $e_N(\mu)$ is

$$e_N(\mu)=\frac{\det(p_{ij}(\mu; i\eta^0))}{\text{G.C.M.}\,(p^*_{ij}(\mu; i\eta^0))}=\frac{Q(\mu; i\eta^0)}{q^*(\mu; i\eta^0)}(\mu-\mu_1(\eta^0))^{\nu_1-\nu},$$

where $q^*(\mu; i\eta^0)\neq 0$ at $\mu=\mu_1(\eta^0)$. If $\nu_1-\nu=0$, then $e_N(\mu)$ will no longer have the factor $(\mu-\mu_1(\eta^0))$, therefore, no $e_j(\mu)$ has the cofactor $(\mu-\mu_1(\eta^0))$. Hence

$$\det(p_{ij}(\mu; i\eta^0))=e_1(\mu)e_2(\mu)\cdots e_N(\mu)$$

has no factor $(\mu-\mu_1(\eta^0))$, which is a contradiction. From the above discussion, there exists $p^*_{ij}(\mu; i\eta^0)$, say $p^*_{11}(\mu; i\eta^0)$, such that

(4.4) $$p^*_{11}(\mu; i\eta^0)=(\mu-\mu_1(\eta^0))^{\nu}q^*_{11}(\mu; i\eta^0).$$

Then we add a lower term to $p_{11}(\mu; i\eta^0)$ as follows;

(4.5) $$p_{11}(\mu; i\eta, |\xi|)=p_{11}(\mu; i\eta)+c(i\eta_l)^s|\xi|^{s-mp},$$

where c is a constant (a complex constant in general), l is an index such that $\eta^0_l\neq 0$, and

(4.6) $$\{m_1(\nu_1-\nu)\}p<s<m_1p.$$ [4)]

Consider the matrix

4) Cf. foot note 3).

$$\begin{bmatrix} P_{11}(\mu; i\eta, |\xi|) & * \\ * & p_{ij}(\mu; i\eta) \end{bmatrix}$$

obtained by replacing only the $p_{11}(\mu; i\eta)$ of the matrix $(p_{ij}(\mu; i\eta))$ with $P_{11}(\mu; i\eta, |\xi|)$.

Now we consider the system of p-evolution equations;

$$\begin{bmatrix} P_{11}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) & * \\ * & p_{ij}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix},$$

where $P_{11}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) = p_{11}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) + c\left(\frac{\partial}{\partial x_l}\right)^s$,

its characteristic equation is equivalent to

$$\det \begin{bmatrix} P_{11}(\mu; i\eta, |\xi|) & * \\ * & p_{ij}(\mu; i\eta) \end{bmatrix} = 0. \tag{4.7}$$

Put $\eta = \eta^0$, and denote $Q(\mu; i\eta^0)$, $q_{11}^*(\mu; i\eta^0)$ by $Q(\mu)$, $q_{11}^*(\mu)$ respectively. Then (4.7) becomes

$$(\mu - \mu_1(\eta^0))^\nu [(\mu - \mu_1(\eta^0))^{\nu_1 - \nu} Q(\mu) + q_{11}^*(\mu) c(i\eta_l^0)^s |\xi|^{s - m_1 p}] = 0.$$

This implies that the characteristic equation (4.7) has the roots of the form;

$$\mu = \mu_1(\eta^0) + \left(-\frac{q_{11}^*(\mu_1(\eta^0)) c(i\eta_l^0)^s}{Q(\mu_1(\eta^0))}\right)^{\frac{1}{\nu_1 - \nu}} |\xi|^{\frac{s - m_1 p}{\nu_1 - \nu}} + o(|\xi|^{\frac{s - m_1 p}{\nu_1 - \nu}}). \tag{4.8}$$

Then, by choosing the argument of the constant c in a suitable way, we see that there exists one root, say $\mu_1^*(\eta^0)$, satisfying

$$\text{real part } \mu_1^*(\eta^0) \geq \delta' |\xi|^{\frac{s - m_1 p}{\nu_1 - \nu}} \tag{4.9}$$

where δ' is a positive constant. Then, this implies that

$$\text{real part } \lambda_1^*(\eta^0 |\xi|) \to +\infty, \text{ when } |\xi| \to +\infty.$$

q.e.d.

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