99. *Ergodic Theorems for Pseudo-resolvents*

By Kôsaku YOSIDA
Department of Mathematics, University of Tokyo

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1. **The theorem.** Let X be a complete locally convex linear topological space, and $L(X, X)$ the algebra of all continuous linear operators on X into X. A pseudo-resolvent J_λ is a function on a subset $D(J)$ of the complex plane with values in $L(X, X)$ satisfying the resolvent equation

(1) $$J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu.$$

We have, denoting by I the identity operator,

(2) $$(I - \lambda J_\lambda) = (I - (\lambda - \mu) J_\lambda)(I - \mu J_\mu)$$

and

(3) $$\lambda J_\lambda (I - \mu J_\mu) = (1 - \mu(\mu - \lambda)^{-1}) \lambda J_\lambda - \lambda(\lambda - \mu)^{-1} \mu J_\mu.$$

We see, by (1), that all J_λ, $\lambda \in D(J)$, have a common null space $N(J)$ and a common range $R(J)$. We also see, by (2), that all $(I - \lambda J_\lambda)$, $\lambda \in D(J)$, have a common null space $N(I-J)$ and a common range $R(I-J)$. $N(J)$ and $N(I-J)$ are closed linear subspace of X, but $R(J)$ and $R(I-J)$ need not be closed; we shall denote by $R(J)^a$ and $R(I-J)^a$ their closures respectively.

To formulate our ergodic theorems we prepare two lemmas.

Lemma 1. Let there exist a sequence $\{\lambda_n\}$ of numbers $\in D(J)$ such that

(4) $\lim_{n\to\infty} \lambda_n = 0$ and the family of operators $\{\lambda_n J_{\lambda_n}\}$ is equi-continuous.

Then we have

(5) $$R(I-J)^a = P(J) = \{x \in X; \lim_{n\to\infty} \lambda_n J_{\lambda_n} x = 0\},$$

and hence

(6) $$N(I-J) \cap R(I-J)^a = \{0\}.$$

Lemma 1′. Let there exist a sequence $\{\lambda_n\}$ of numbers $\in D(J)$ such that

(4)′ $\lim_{n\to\infty} |\lambda_n| = \infty$ and the family of operators $\{\lambda_n J_{\lambda_n}\}$ is equi-continuous.

Then we have

(5)′ $$R(J)^a = I(J) = \{x \in X; \lim_{n\to\infty} \lambda_n J_{\lambda_n} x = x\}$$

and hence

(6)′ $$N(J) \cap R(J)^a = \{0\}.$$

Our ergodic theorems read as follows.

Theorem 1. Let (4) be satisfied. Let, for a given $x \in X$, there exist a subsequence $\{\lambda_{n'}\}$ of $\{\lambda_n\}$ such that

(7) $$\text{weak-}\lim_{n'\to\infty} \lambda_{n'} J_{\lambda_{n'}} x = x_h \text{ exists.}$$

Then $x_h = \lim_{n\to\infty} \lambda_n J_{\lambda_n} x$ and $x_h \in N(I-J)$, $x_p = x - x_h \in P(J)$.

Corollary 1. Let (4) be satisfied, and let X be locally sequentially weakly compact. Then

(8) $$X = N(I-J) \oplus R(I-J)^a = N(I-J) \oplus P(J) \quad \text{(direct sum).}$$

Theorem 1'. Let (4)' be satisfied. Let, for a given $x \in X$, there exist a subsequence $\{\lambda_{n'}\}$ of $\{\lambda_n\}$ such that

(7)' $$\text{weak-}\lim_{n'\to\infty} \lambda_{n'} J_{\lambda_{n'}} x = x_{h'} \text{ sxists.}$$

Then $x_{h'} = \lim_{n\to\infty} \lambda_n J_{\lambda_n} x$ and $x_{h'} \in I(J)$, $x_{p'} = x - x_{h'} \in N(J)$.

Corollary 1'. Let (4)' be satisfied, and let X be locally sequentially weakly compact. Then

(8)' $$X = N(J) \oplus R(J)^a = N(J) \oplus I(J) \quad \text{(direct sum).}$$

Remark. The pseudo-resolvent J_λ is a resolvent of a closed linear operator A if and only if $N(J)=0$; in this case $R(J)$ coincides with the domain $D(A)$ of A. When J_λ is the resolvent of a closed linear operator A with domain $D(A)$ and range $R(A)$ both in a Banach space X, the Theorem 1 and Theorm 1' respectively correspond to the "abelian ergodic theorem at ∞ for semi-groups" and the "abelian ergodic theorem at 0 for semi-groups". Our formulation is more general than those due to E. Hille and R.S. Phillips.[1] The Theorem 1' for the case of a Banach space X is due to T. Kato,[2] and our formulation of two theorems is modelled after him.

2. Proofs of the theorems.

Proof of Lemma 1. We see, by (3) and (4), that $x \in R(I-J)$ implies $x \in P(J)$. Let $y \in R(I-J)^a$. Then, for any semi-norm q on X and $\varepsilon > 0$, there exists $x \in R(I-J)$ such that $q(y-x) < \varepsilon$. By (4), we have, for any semi-norm q' on X, $q'(\lambda_n J_{\lambda_n}(y-x)) \leqq M q(y-x)$ where the positive constant M depends on q and q'. Thus we see that y must belong to $P(J)$.

Let conversely $x \in P(J)$. Then, for any semi-norm q on X and $\varepsilon > 0$, there exists λ_n such that $q(x - (x - \lambda_n J_{\lambda_n} x)) < \varepsilon$. Hence x must belong to $R(I-J)^a$.

The proof of Lemma 1' may be obtained similarly by making use of (3) and (4)'.

Proof of Theorem 1. Setting $\mu = \lambda_{n'}$ in (2) and letting $n' \to \infty$, we see, by (4), that $x_h \in N(I-J)$. We have thus

(9) $$\lambda_n J_{\lambda_n} x = x_h + \lambda_n J_{\lambda_n}(x - x_h),$$

and hence we have to prove that $(x - x_h) \in P(J)$. We prove it from

1) Functional analysis and semi-groups, Providence, 502 (1957).

2) Remarks on pseudo-resolvents and infinitesimal generators of semi-groups, Proc. Japan Acad. **35**, 467-468 (1959).

Lemma 1 and $(x-\lambda_{n'}J_{\lambda_{n'}}x)\in R(I-J)$, observing that, in a locally convex linear topological space X, any closed linear subspace is weakly closed.

Proof of Theorem 1'. Setting $\mu=\lambda_{n'}$ in (3) and letting $n'\to\infty$, we see, by (4)', that $\lambda J_\lambda(x-x_{h'})=0$, that is, $x_{p'}=x-x_{h'}\in N(J)$. On the other hand, we have $x_{h'}\in R(J)^a$. For, $\lambda_{n'}J_{\lambda_{n'}}x\in R(J)$ and the closed linear subspace $R(J)^a$ is weakly closed. Thus, by Lemma 1', $x_{h'}\in I(J)$.

3. Some applications.

i) *An application to fractional powers of closed operators.* Let J_λ be the resolvent $(\lambda I-A)^{-1}$ of a closed linear operator A and $D(J)=\{\lambda;\lambda>0\}$. If we assume that

(4)'' the family of operators $\{\lambda(\lambda I-A)^{-1};\ \lambda>0\}$ is equi-continuous, then the fractional power A^α of A, $0<\alpha<1$, may be defined as the smallest closed extension of the operator

$$A^\alpha x=\frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{\alpha-1}(\lambda I-A)^{-1}(-Ax)d\lambda \text{ for } x\in D(A).\text{[3]} \tag{10}$$

Then the equation

$$\lim_{\alpha\downarrow 0} A^\alpha x=x \tag{11}$$

is satisfied for $x\in D(A)\cap P(J)$. This we see from the fact that such an x satisfies the "Poisson equation"

$$\lim_{\alpha\downarrow 0}(\lambda I-A)^{-1}(-Ax)=x. \tag{12}$$

ii) *An application to potential theory.* Suggested by a special case when J_λ is the resolvent of the Laplacian, we may call the elements of $N(I-J)$ "harmonic" and the elements of $P(J)$ "potentials." Then the Theorem 1 may be considered as an analogue of F. Riesz decomposition for subharmonic functions. The analogy is more close if we introduce the notion of "subharmonicity". To this end, we assume that a notion of "positivity", denoted by $x\geqq 0$, is defined in X in such a way that X is a semi-ordered linear space satisfying the condition:

(13) a monotone increasing bounded sequence of elements $\in X$ converges weakly to an element of X which is greater than the elements of the sequence.

We further assume that $D(J)$ contains an open interval $(0,\lambda_0)$ and the operator J_λ is positive for $\lambda\in(0,\lambda_0)$ in the sense that $x\geqq 0$ implies $J_\lambda x\geqq 0$. We shall call an element $x\in X$ "subharmonic" if it satisfies

$$\lambda J_\lambda x\geqq x \text{ for some and hence for all } \lambda\in(0,\lambda_0). \tag{14}$$

Then we can prove the following corollary of Theorem 1.

3) A. M. Balakrishnan: Fractional powers of closed operators and semi-groups generated by them, Pacific J. of Math.; K. Yosida: Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them, Proc. Japan Acad. **36**, 86–89 (1960); T. Kato: Note on fractional powers of linear operators, Ibid., 94–96.

Corollary 2. Let

(15) the family of operators $\{\lambda J_\lambda; 0<\lambda<\lambda_0\}$ is equi-continuous.

Then a "subharmonic" element x is uniquely decomposed as the sum of a "harmonic" element x_h and a "potential" x_p. The "harmonic" part x_h of x is given by $x_h=\lim_{\lambda\downarrow 0}\lambda J_\lambda x$ and x_h is the "least harmonic majorant" of x.

Proof. By (2) and the positivity of J_λ, we see that

(16) $$\lambda<\mu \text{ implies } \lambda J_\lambda x \geqq \mu J_\mu x \geqq x.$$

Therefore, by (13) and (12), weak-$\lim_{\lambda\downarrow 0}\lambda J_\lambda x=x_h$ exists, and the first part of the Corollary is proved.

Let a "harmonic" element x_H satisfy $x_H\geqq x$. Then, by the positivity of λJ_λ and the "harmonicity" of x_H, we have

$$x_H=\lambda J_\lambda x_H \geqq \lambda J_\lambda x \text{ and hence } x_H\geqq x_h.$$

iii) *An application to semi-group theory.* The following corollaries of Theorem 1′ are obtained after T. Kato, loc. cit. in the footnote 2.

Corollary 2′. If J_λ is a pseudo-resolvent satisfying (4)′ and if $R(J)$ is dense in X, then $N(J)=\{0\}$ and hence J_λ is a resolvent.

Corollary 3′. If X is locally sequentially weakly compact and J_λ is the resolvent of a closed linear operator A satisfying (4)′, then the domain $D(A)$ of A is dense in X.