# 122. Further Measure-Theoretic Results in Curve Geometry

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1. **Extension of a previous result.** The final theorem of our recent note [4] was only given a sketched proof. We shall now prove it completely, extending it at the same time to the following form which is slightly more general.

THEOREM. *If $I$ is an interval (of any type) on which a curve $\varphi$, situated in $\boldsymbol{R}^m$, is both continuous and rectifiable, then $\Xi(\varphi; E) = \Gamma(\varphi; E)$ for any subset $E$ of $I$ whatsoever.*

PROOF. We may clearly suppose $I$ an endless interval, so that $\varphi$ is continuous at all points of $I$. Let us denote by $\mathfrak{N}$ the class of all the Borel sets $X \subset I$ fulfilling the relation $\Xi(\varphi; X) = \Gamma(\varphi; X)$, and by $\mathfrak{M}$ the class of all convex subsets of $I$. Each set $(\mathfrak{M})$ being then either void, or a one-point set, or an interval, we see as at the end of [4] that the class $\mathfrak{N}$ contains $\mathfrak{M}$. As may be readily verified further, $\mathfrak{M}$ is a primitive class in $I$ (see p. 116 of our paper [1] for the terminology). In other words, $\mathfrak{M}$ satisfies the following three conditions: (i) the interval $I$ belongs to $\mathfrak{M}$; (ii) if $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$, then $AB \in \mathfrak{M}$; (iii) if $A \in \mathfrak{M}$, there is a disjoint infinite sequence $\varDelta$ of sets $(\mathfrak{M})$ such that $I - A = [\varDelta]$. Consequently, in conformity with Theorem 1 of [1], the smallest additive class (in $I$) containing the class $\mathfrak{M}$ coincides with the smallest normal class containing $\mathfrak{M}$ (see Saks [7], p. 83, for the terminology). But, taking into account the rectifiability of $\varphi$ on $I$, we find easily that $\mathfrak{N}$ is a normal class. It follows at once that $\mathfrak{N}$ coincides with the Borel class in $I$, so that our assertion holds at least whenever $E$ is a Borel subset of $I$.

Let us turn now to the case of general $E$. As it will follow from the lemma to be soon established below, we can enclose $E$ in a Borel set $E_0 \subset I$ such that $\Gamma(\varphi; E) = \Gamma(\varphi; E_0)$. Since $\Gamma(\varphi; E_0) = \Xi(\varphi; E_0)$ by what has already been proved, we obtain $\Gamma(\varphi; E) \geqq \Xi(\varphi; E)$. This, combined with the lemma of [4]§2, gives finally $\Gamma(\varphi; E) = \Xi(\varphi; E)$.

LEMMA. *If a curve $\varphi$ is continuous at all points of a set $E$, we can enclose $E$ in a set $H$ of the class $\mathfrak{G}_\delta$ such that $\Gamma(\varphi; H) = \Gamma(\varphi; E)$.*

PROOF. We may plainly assume $\Gamma(\varphi; E)$ finite. To simplify our notations, let us write $\Phi(X) = \mathrm{d}(\varphi[X])$ for each set $X$. Given any natural number $n$, the set $E$ has an expression as the join of an infinite sequence $\varDelta_n = \langle X_1^{(n)}, X_2^{(n)}, \cdots \rangle$ of its subsets such that $\mathrm{d}(X_i^{(n)}) < \varepsilon$ for $i = 1, 2, \cdots$ and $\Phi(\varDelta_n) < \Gamma(\varphi; E) + \varepsilon$, where and below we write $\varepsilon = n^{-1}$

for convenience. By hypothesis we can associate with $\Delta_n$ an infinite sequence of open sets, $\Theta_n=\langle G_1^{(n)}, G_2^{(n)},\cdots\rangle$, such that for each $i$ the set $G_i^{(n)}$ contains $X_i^{(n)}$, has diameter less than $\varepsilon$, and fulfils $\Phi(G_i^{(n)})<\Phi(X_i^{(n)})+2^{-i}\varepsilon$. Writing $H=[\Theta_1][\Theta_2]\cdots$ we see at once that $H$ is a set $(G_\delta)$ containing $E$. We have further, for each positive integer $n$,

$$\Gamma_\varepsilon(\varphi; H)\leqq\Gamma_\varepsilon(\varphi; [\Theta_n])\leqq\Phi(\Theta_n)<\Phi(\Delta_n)+\varepsilon<\Gamma(\varphi; E)+2\varepsilon,$$

where $\Gamma_\varepsilon$ has the same meaning as in [4]§2. Recalling $\varepsilon=n^{-1}$ and making $n\to+\infty$, we find in the limit $\Gamma(\varphi; H)\leqq\Gamma(\varphi; E)$. Since the converse inequality is obvious, this completes the proof.

REMARK. The lemma just established may be slightly further generalized as follows. *If $A$ is a nonvoid set containing a set $E$ and if the subcurve $(\varphi; A)$ of $\varphi$ is continuous at all points of $E$, then $E$ can be enclosed in a set $H$ which is relatively $\mathfrak{G}_\delta$ with respect to $A$ and such that $\Gamma(\varphi; H)=\Gamma(\varphi; E)$.* The above proof remains valid for this also, provided a few verbal changes are made.

**2. Countable rectifiability of a curve on a set.** We shall call a curve $\varphi$ to be *countably rectifiable* on a set $E$ iff $E$ can be expressed as the join of a (finite or infinite) sequence of its subsets on each of which $\varphi$ is rectifiable. It is obvious that this is the case when and only when all the coordinate-functions of $\varphi$ are VBG on $E$, i.e. of generalized bounded variation on $E$ (*vide* Saks [7], p. 221). When especially $E$ coincides with the whole real line, the reference to the set $E$ will usually be omitted and we shall simply say that $\varphi$ is a *countably rectifiable curve*. This being so, the theorem of the preceding section admits of the following extension.

THEOREM. *If $\varphi$ is rectifiable on a set $E$, then $\Xi(\varphi; E)=\Gamma(\varphi; E)$. The same conclusion holds also when $\varphi$ is countably rectifiable on $E$, provided that $\varphi$ is continuous on the same set.*

PROOF. 1) In order to prove the first half of the assertion, we may suppose without loss of generality that $\varphi$ is rectifiable on the whole line $\boldsymbol{R}$. For, according to Lemma (4.1) on p. 221 of Saks [7], each coordinate-function of the curve $\varphi$ coincides on $E$ with a function which is of bounded variation on $\boldsymbol{R}$. If we now argue as in the proof for the theorem of [4]§4, we can easily construct a strictly increasing function $q(t)$ and a rectifiable continuous curve $\omega(u)$, such that $\omega(q(t))=\varphi(t)$ for every point $t\in\boldsymbol{R}$. This allows us to assume $\varphi$ both rectifiable and continuous, and the assertion then follows directly from the theorem of §1.

2) Supposing $\varphi$ continuous on $E$ as well as countably rectifiable on $E$, let us express $E$ as the join of an infinite sequence $E_1, E_2,\cdots$ of sets on each of which $\varphi$ is rectifiable. Then we have the evident relation $L(\varphi; E\overline{E}_n)=L(\varphi; E_n)<+\infty$ for each $n$, the bar indicating the closure operation. Consequently it follows from what has just been

proved in part 1) that, if we write $H_1=\overline{E}_1$, and $H_n=\overline{E}_n-(\overline{E}_1\smile\cdots\smile\overline{E}_{n-1})$ when $n>1$, then $\Xi(\varphi; EH_n)=\Gamma(\varphi; EH_n)$ for each $n=1,2,\cdots$. On the other hand, noting that $H_1, H_2,\cdots$ constitute a disjoint sequence of Borel sets and making use of Theorem (4.6) on p. 46 of Saks [7], we derive $\Xi(\varphi; E)=\sum\Xi(\varphi; EH_n)$, the sum extending over all $n$, together with a similar relation for $\Gamma$. The above results, collected together, lead at once to the required equality and complete the proof of the theorem.

3. **Multiplicity functions.** Given a curve $\varphi$ (in $\boldsymbol{R}^m$) and a set $E$, we call *multiplicity function* determined by $\varphi$ and $E$, and denote by the symbol $N(\varphi; x; E)$, the function defined for each point $x$ of $\boldsymbol{R}^m$ to be the number (finite or $+\infty$) of the points $t$ of $E$ such that $\varphi(t)=x$. Sometimes this is also termed *multiplicity* of $x$ with respect to $\varphi$ and $E$. We may write for this simply $N(x; E)$, or more concisely $N(x)$, when there is no fear of ambiguity.

In the present section we shall only consider the case in which the space $\boldsymbol{R}^m$ is the real line, so that $\varphi$ reduces itself to a function. It would not be uninteresting, however, to extend to the case of curves the results that we are going to establish in the sequel; the paper [5] by Nöbeling, which is inaccessible to the author at present, would probably be useful for that purpose.

We find it convenient to begin with the following lemma which, in view of the first theorem of [4] §5, generalizes the proposition III. 2.25 of Radó [6].

LEMMA. *If a function $f(t)$ is of bounded variation on a Borel set $E$, then the multiplicity function $N(f; x; E)$ is summable on $\boldsymbol{R}$ (in the Lebesgue sense) and we have*

$$\Xi(f; E)=\Gamma(f; E)=\int_{\boldsymbol{R}} N(f; x; E)\, dx.$$

*In particular, therefore, the image $f[E]$ is a measurable set.*

PROOF. On account of Lemma (4.1) on p. 221 of Saks [7], the function $f$ may be assumed VB on the whole $\boldsymbol{R}$. Then the set $E_0$ of the points of $E$ at which $f$ is discontinuous must be countable, and hence the conclusion of the assertion holds if we replace in it the set $E$ by $E_0$ throughout. On the other hand $N(f; x; X)$, $\Xi(f; X)$, and $\Gamma(f; X)$, *qua* functions of a variable Borel set $X$, are evidently additive. Thus we need only consider the case where $E$ is a nonvoid Borel set at all points of which $f$, assumed VB on $\boldsymbol{R}$, is continuous. We now argue as in the proof for the theorem of [4]§4 and construct a strictly increasing function $q(t)$ and a continuous function $F(u)$ which is VB on $\boldsymbol{R}$, in such a manner that $F(q(t))=f(t)$ for every $t\in\boldsymbol{R}$ and further that the inverse function $q^{-1}(u)$ of $q$, defined for the points $u$ of the set $q[\boldsymbol{R}]$, coincides on the set $E^*=q[E]$ with some continuous non-decreasing function $p(u)$ defined on $\boldsymbol{R}$ and with the property $E^*=p^{-1}[E]$. Then $E^*$ is plainly a Borel set and we find moreover

$$\Xi(f;E)=\Xi(F;E^*),\ \Gamma(f;E)=\Gamma(F;E^*),\ N(f;x;E)=N(F;x;E^*),$$

the last relation holding for all $x\in\boldsymbol{R}$.

On gathering all that has been said in the above, it follows immediately that we are allowed to confine ourselves to the case where the function $f$ is VB on $\boldsymbol{R}$ and continuous everywhere. This being so, let us denote by $\mathfrak{N}$ the class of all the Borel sets $E$ for which our assertion is true, and by $\mathfrak{M}$ the primitive class consisting of all convex sets in $\boldsymbol{R}$. With the combined help of the first theorem in [4]§5 and of Theorem (6.4) on p. 280 of Saks [7], which is easily seen to hold even when $I_0$ is an interval of arbitrary type, we then see at once that the class $\mathfrak{N}$ contains $\mathfrak{M}$. Furthermore it is verified without difficulty that $\mathfrak{N}$ is a normal class. Accordingly we conclude as in §1 that $\mathfrak{N}$ must coincide with the Borel class in $\boldsymbol{R}$. This completes the proof.

THEOREM. *If a function $f(t)$ is continuous on a Borel set $E$ and at the same time of generalized bounded variation on $E$, then the multiplicity function $N(f;x;E)$ is measurable and we have*

$$\Xi(f;E)=\Gamma(f;E)=\int_{\boldsymbol{R}} N(f;x;E)\,dx.$$

*In consequence, the image $f[E]$ must be a measurable set.*

PROOF. As in part 2) of the proof for the theorem of §2, we can decompose $E$ into an infinite sequence of Borel sets $B_1, B_2,\cdots$ such that $L(f;B_n)<+\infty$ for $n=1,2,\cdots$. Then the above lemma shows that $N(x;B_n)$ is a measurable function of $x$ for each $n$ and that the relation of our theorem holds when we replace in it the set $E$ by $B_n$ throughout. The function $N(x;E)$, which equals $N(x;B_1)+N(x;B_2)+\cdots$, is therefore measurable also and the assertion follows readily on integrating the last series term-by-term.

**4. Saltus- and continuous parts of a locally rectifiable curve.** Let $\psi(I)$ be an *interval-curve* situated in $\boldsymbol{R}^m$, that is to say, an $m$-tuple of additive interval-functions defined for linear closed intervals $I$ and assuming finite real values. If $\varphi(t)$ is a curve which corresponds to $\psi(I)$ in the sense that $\varphi(I)=\psi(I)$ for each $I$, then the reduced and Hausdorff measure-lengths of $\varphi$ on a set $X$ are independent of the choice of $\varphi$. For evidently any two copies of $\varphi$ can only differ by a parallel translation. We are thus entitled to write $\Xi(\psi;X)$ and $\Gamma(\psi;X)$ for these two quantities and to term them *reduced* and *Hausdorff measure-lengths* of $\psi$ over $X$ respectively. We can further define the *length* $L(\psi;X)$ and the *measure-*

*length* $L_*(\psi; X)$ similarly.

In our recent work [3] we gave to the notion of saltus-functions a new definition, different from that in Saks [7] and valid for any Euclidean space. Now, by the *saltus-* and *continuous parts* of a locally rectifiable curve $\varphi(t)$ [or of a point-function $f(t)$ of locally bounded variation] we shall understand the corresponding quantities, in the sense of §49 [or §9] of [3], for the interval-curve [or the interval-function] determined by $\varphi$ [or by $f$]. Thus the saltus-part of $\varphi$ is an interval-curve consisting of the saltus-parts of the coordinate-functions of $\varphi$, and similarly for the continuous part of $\varphi$.

THEOREM. *If $\psi(I)$ is the saltus-part of a locally rectifiable curve $\varphi(t)$, then $L_*(\psi; E)=0$ for each set $E$ at all points of which $\varphi$ is continuous. Further, the reduced (and therefore also the Hausdorff) measure-length of $\psi$ vanishes identically.*

PROOF. It is sufficient to establish the first half of the assertion. In fact we have $\Xi(\psi; E)=L_*(\psi; E)$ by the theorem of [4]§4, while on the other hand $\Xi(\psi; X)$ vanishes for every countable set $X$ (see [4]§2) and in particular for any set consisting of points of discontinuity for $\varphi$. Writing for short $s(I)=L(\psi; I)$, where $I$ is any closed interval, we observe that $s(I)$ is a finite-valued additive interval-function whose points of continuity coincide with those of the interval-function $\psi(I)$, i.e. of the curve $\varphi(t)$. Since moreover $s^*(X)=L_*(\psi; X)$ for every set $X$ as remarked in [2]§4, our task comes to proving the relation $s^*(E)=0$ for each set $E$ at whose points $s(I)$ is continuous. Of course we may assume $E$ bounded, i.e. contained in some closed interval $K$.

Now $s(I)$, being the length of the saltus-part of $\varphi$, must itself be a saltus-function, that is to say, coincides with its own saltus-part (*vide* [3], §12 and §49). Accordingly $s(K)$ is the supremum of the sum $s_0(\Delta)$, where $\Delta$ is an arbitrary subdivision of $K$ into a finite number of closed intervals and where $s_0(I)$ denotes further, for each closed interval $I$, the inside excess of $s$ over $I$ (*vide* [3], §1 and §3). But if $I^\circ$ stands for the interior of $I$, we have $s_0(I)=s(I)-s^*(I^\circ)$. In point of fact, $s$ is a nonnegative additive function and so $s^*(I^\circ)$ is the supremum of $s(J)$ for closed intervals $J\subset I^\circ$. It follows directly that, given any positive number $\varepsilon$, there exists in $K$ a finite set $M$ containing both the extremities of $K$ and subject to the condition $s^*(K-M)<\varepsilon$. Noting the evident inclusion $E\subset(K-M)\smile(EM)$, where $s^*(EM)=0$ since $s$ is continuous at all points of $E$ by assumption, we deduce at once that $s^*(E)\leqq s^*(K-M)<\varepsilon$. On making $\varepsilon\to 0$, this gives $s^*(E)=0$, completing the proof.

THEOREM. *If $\omega$ is the continuous part of a locally rectifiable curve $\varphi(t)$, then $\Gamma(\varphi; E)=\Xi(\varphi; E)=\Xi(\omega; E)=\Gamma(\omega; E)$ for each set $E$.*

PROOF. The theorem of §2, together with Theorem (4.6) on p. 46 of Saks [7], implies that it is enough to derive the middle equality concerning the reduced measure-lengths. By arguing as in the proof of the preceding theorem, this is further reduced to proving the relation $L_*(\varphi; A)=L_*(\omega; A)$ for each set $A$ at whose points the curve $\varphi$ is continuous. Let us now write $\psi$ for the saltus-part of $\varphi$ as above, so that $\varphi(I)=\omega(I)+\psi(I)$ for every closed interval $I$. We then have $L_*(\varphi; X)\leqq L_*(\omega; X)+L_*(\psi; X)$ for any set $X$, as is easily verified by considering successively the three cases in which $X$ is respectively an endless interval, an open set, and a general set (cf. the theorem of [2]§4). It follows at once, in view of the foregoing theorem, that $L_*(\varphi; A)\leqq L_*(\omega; A)$. Similarly we deduce that $L_*(\omega; A)\leqq L_*(\varphi; A)$, starting this time from the relation $\omega(I)=\varphi(I)-\psi(I)$. Consequently we get $L_*(\varphi; A)=L_*(\omega; A)$, which completes the proof.

## References


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