7. On Adjunction Spaces

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1. **The Main Theorem.** Let $\{C_\alpha \mid \alpha \in \Omega\}$ be a family of topological spaces. Let us consider a family of continuous maps $\{g_\alpha \mid \alpha \in \Omega\}$, where g_α is a continuous map defined on a *closed* subspace A_α of C_α into another topological space Y for each α. Then the *disjoint union* $W = Y \smile (\underset{\alpha \in \Omega}{\smile} C_\alpha)$ is a space with the topology defined as follows: a subset $V \subset W$ is open if and only if $V \frown Y$ is an open subset of Y and $V \frown C_\alpha$ is an open subset of C_α for each α. Now we define in W an equivalence relation as follows: Two points $x \in C_\alpha$ and $y \in Y$ are equivalent if and only if $g_\alpha(x) = y$; two points $x \in C_\alpha$ and $y \in C_\beta$ are equivalent if and only if $g_\alpha(x) = g_\beta(y)$; each point is equivalent to itself. We take Z to be the quotient space of W with respect to this equivalence relation and $p: W \to Z$ the natural projection; that is, a subset B of Z is open if and only if $p^{-1}(B)$ is an open subset in W. We call this space Z the *adjunction space obtained by adjoining $\{C_\alpha\}$ to Y by means of the continuous maps $\{g_\alpha : A_\alpha \to Y\}$.*

The adjunction space is one of the most important spaces in the homotopy theory. (Cf. Hu [1].) We shall consider here a set-theoretical property of this space. Namely we shall prove the following theorem.

Theorem 1. *Let $\{C_\alpha \mid \alpha \in \Omega\}$ be a family of topological spaces, and let A_α be a closed subspace of C_α, g_α a closed continuous map defined on A_α into another topological space Y, for each $\alpha \in \Omega$. Then each of the following properties for Y and all C_α's, implies the same property for the adjunction space Z, obtained by adjoining $\{C_\alpha\}$ to Y by means of the continuous maps $\{g_\alpha : A_\alpha \to Y\}$:*

(1) *normality,* (2) *complete normality,*
(3) *perfect normality,* (4) *collectionwise normality,*
(5) $\mathfrak{m}$-*paracompactness and normality,*

where $\mathfrak{m}$ is any infinite cardinal number.

Here a topological space is called $\mathfrak{m}$-*paracompact* if any open covering of power $\leqq \mathfrak{m}$ admits a locally finite open refinement. This notion is due to K. Morita [3].

In his lecture on the obstruction theory of CW-complexes [4], G. W. Whitehead has introduced the notion of relative CW-complexes. (For the definition, see §3 below.) As an application of Theorem 1, we shall establish the following theorem.

Theorem 2. *Any relative CW-complex (X, Y) has one of the following properties if and only if Y has the same property*:

(1) *normality*, (2) *complete normality*,

(3) *perfect normality*, (4) *collectionwise normality*,

(5) $\mathfrak{m}$-*paracompactness and normality*,

where $\mathfrak{m}$ *is any infinite cardinal number.*

In particular, any CW-complex ([4]) is a paracompact and normal space. (Cf. K. Morita [2].)

2. Proof of Theorem 1. Lemma 1. *If we put $g'_\alpha = p \mid C_\alpha : C_\alpha \to Z$ (i.e. the restriction of the continuous map p to C_α) for each $\alpha \in \Omega$, and put $g' = p \mid Y : Y \to Z$, then g' and each g'_α are closed continuous maps respectively.*

Proof. It is obvious that g' is a closed continuous map. To prove that g'_α is a closed continuous map, it is sufficient to show that, for any closed subset A of C_α, $p^{-1}(g'_\alpha(A))$ is a closed subset of W. Since g_α is a closed continuous map and $p^{-1}(g'_\alpha(A)) \frown Y = g_\alpha(A \frown A_\alpha)$, $p^{-1}(g'_\alpha(A)) \frown Y$ is a closed subset of Y. Since $p^{-1}(g'_\alpha(A)) \frown C_\alpha = A \smile g_\alpha^{-1}(g_\alpha(A \frown A_\alpha))$, $p^{-1}(g'_\alpha(A)) \frown C_\alpha$ is a closed subset of C_α. Finally, for any C_β, $\beta \neq \alpha$, $p^{-1}(g'_\alpha(A)) \frown C_\beta = g_\beta^{-1}(g_\alpha(A \frown A_\alpha))$, and since g_α is a closed continuous map, $p^{-1}(g'_\alpha(A)) \frown C_\beta$ is a closed subset of C_β. Therefore $p^{-1}(g'_\alpha(A))$ is a closed subset of W, and our lemma is established.

K. Morita has introduced the following notion in [2] (also in [3]). Let X be a topological space and $\{A_\alpha \mid \alpha \in \Omega\}$ be a closed covering of X. Then X is said to *have the weak topology with respect to* $\{A_\alpha\}$, if the union of any subcollection $\{A_\beta \mid \beta \in \Lambda\}$ of $\{A_\alpha\}$ is closed in X, and any subset of $\smile_{\beta \in \Lambda} A_\beta$, whose intersection with each A_β is open relative to the subspace topology of A_β, is necessarily open in the subspace $\smile_{\beta \in \Lambda} A_\beta$.

Lemma 2. *The adjunction space Z has the weak topology with respect to the closed covering $\{g'_\alpha(C_\alpha) \smile g'(Y) \mid \alpha \in \Omega\}$.*

Proof. By Lemma 1, g' and each g'_α are closed continuous maps, and hence $\{g'_\alpha(C_\alpha) \smile g'(Y) \mid \alpha \in \Omega\}$ is a closed covering of Z.

We must show that, for any subset Λ of Ω, any subset A of $\smile_{\beta \in \Lambda}\{g'_\beta(C_\beta) \smile g'(Y)\}$, whose intersection with $g'_\beta(C_\beta) \smile g'(Y)$ is a closed subset of $g'_\beta(C_\beta) \smile g'(Y)$, is necessarily a closed subset of Z.

Since $A \frown (g'_\beta(C_\beta) \smile g'(Y))$ is closed by assumption and $A \frown g'(Y) = [A \frown (g'_\beta(C_\beta) \smile g'(Y))] \frown g'(Y)$, $A \frown g'(Y)$ is closed. Hence $p^{-1}(A) \frown Y$ is a closed subset of Y.

For any C_β, $\beta \in \Lambda$, $A \frown (g'_\beta(C_\beta) \smile g'(Y))$ is closed by assumption and $A \frown g'_\beta(C_\beta) = [A \frown (g'_\beta(C_\beta) \smile g'(Y))] \frown g'_\beta(C_\beta)$, and so $A \frown g'_\beta(C_\beta)$ is closed. Hence $p^{-1}(A) \frown C_\beta$, $\beta \in \Lambda$, is a closed subset of C_β.

Finally, for any C_γ, $\gamma \notin \Lambda$, $p^{-1}(A) \frown C_\gamma = g'^{-1}_\gamma(A \frown g'_\gamma(C_\gamma))$, and since $(A \frown g'_\gamma(C_\gamma)) \subset g'_\gamma(A_\gamma) \subset g'(Y)$, we have $A \frown g'_\gamma(C_\gamma) = (A \frown g'_\gamma(C_\gamma)) \frown g'_\gamma(A_\gamma)$

$=(A\frown g'_\gamma(A_\gamma))\frown g'(Y)=g'_\gamma(A_\gamma)\frown[A\frown(g'(Y)\smile g'_\beta(C_\beta))]$ and hence $A\frown g'_\gamma(C_\gamma)$ is closed. Hence $p^{-1}(A)\frown C_\gamma$, $\gamma\notin\Lambda$, is a closed subset of C_γ.

Therefore $p^{-1}(A)$ is a closed subset of W, and so A is a closed subset of Z by the definition. Our lemma is thus established.

Since each of the properties (1)–(5) in Theorem 1 is preserved by a closed continuous map, each subspace $g'_\alpha(C_\alpha)\smile g'(Y)$, $\alpha\in\Omega$, has the same property as C_α and Y. Hence Theorem 1 is obtained by Lemma 2 and the following theorem due to K. Morita [3].

Theorem. *If a topological space X has the weak topology with respect to a closed covering $\{A_\alpha\}$ such that each set A_α is $\mathfrak{m}$-paracompact and normal, then X is $\mathfrak{m}$-paracompact and normal.*

3. **Relative CW-complexes.** We now recall the notion of relative CW-complexes introduced by G. W. Whitehead.

Let X be a Hausdorff space, and Y its closed subspace. If a family of closed subsets $\{E^n_\alpha\mid\alpha\in J_n,\ n=0,1,2,\cdots\}$ satisfies the following conditions, then the family $\{E^n_\alpha\}$ is said to be a *CW-decomposition* of (X,Y), and (X,Y) is called a *relative CW-complex*: If we put $X^n=Y\smile(\bigcup_{m\leqq n}\bigcup_{\alpha\in J_m}E^m_\alpha)\quad(n\geqq 0)$, $X^{-1}=Y$, $\dot{E}^n_\alpha=E^n_\alpha\frown X^{n-1}\quad(n\geqq 0)$, $\operatorname{Int}E^n_\alpha=E^n_\alpha-\dot{E}^n_\alpha$ $(n\geqq 0)$, then

1) $\{\operatorname{Int}E^n_\alpha\mid\alpha\in J_n,\ n=0,1,2,\cdots\}$ is a family of mutually disjoint sets;

2) $X-Y=\bigcup_n\bigcup_{\alpha\in J_n}\operatorname{Int}E^n_\alpha$;

3) for each E^n_α, there exists a continuous map $f^n_\alpha:(I^n,\partial I^n)\to(E^n_\alpha,\dot{E}^n_\alpha)$ such that

i) $f^n_\alpha(I^n)=E^n_\alpha$,

ii) f^n_α, restricted to $\operatorname{Int}I^n$, is a homeomorphism,

where I^n, ∂I^n, $\operatorname{Int}I^n$ denote the n-cube, its usual boundary, its usual interior, respectively;

4) each $\dot{E}^n_\alpha$ intersects with only a finite number of the members of the family $\{\operatorname{Int}E^q_\beta\mid\beta\in J_q,\ q=0,1,2,\cdots\}$;

5) a subset A of X is closed if and only if $A\frown Y$ is a closed subset of Y and $A\frown E^n_\alpha$ is a closed subset of E^n_α for each E^n_α.

We recall also the notion of inductive limit spaces. Let $Y_1\subset Y_2\subset\cdots\subset Y_n\cdots$ be a sequence of topological spaces. Then $Y=\bigcup_n Y_n$ is called the *inductive limit space* of this sequence $\{Y_n\}$ if the topology of Y is defined as follows: a subset V of Y is open if and only if $V\frown Y_n$ is an open subset of Y_n for each n.

Lemma 3. *Let (X,Y) be a relative CW-complex. Then, each subspace X^n, $n=1,2,\cdots$, is the adjunction space obtained by adjoining $\{I^n_\alpha\mid\alpha\in J_n\}$ to X^{n-1} by means of the continuous maps $\{f^n_\alpha\mid\partial I^n:\partial I^n_\alpha\to X^{n-1}\}$, where each I^n_α is a copy of I^n. Moreover, the space X is the inductive limit space of the sequence $Y\subset X^0\subset X^1\subset\cdots\subset X^n\subset\cdots$.*

Proof is omitted.

Proof of Theorem 2. By Lemma 3 and Theorem 1, each subspace X^n has the same property as the subspace Y. Then, the inductive limit space X also has the same property by the following theorem due to K. Morita [3].

Theorem. *If a topological space X has a countable closed covering $\{A_i \mid i=1,2,\cdots\}$ such that any subset C for which $C\frown A_i$ is closed for each i is necessarily closed in X, and if each A_i is $\mathfrak{m}$-paracompact and normal, then X is $\mathfrak{m}$-paracompact and normal.*

Thus the "if" part is established.

As Y is a closed subspace of X, the "only if" part is obvious, and hereby Theorem 2 is established.

References

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[4] G. W. Whitehead: Homotopy theory, Lecture notes at the Massachusetts Institute of Technology (1953).