2. Some Characterizations of Fourier Transforms. III

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1. In this paper we shall denote with $\mathfrak{P}$ the space of all functions on the real number field of class C^∞ whose derivatives decrease rapidly and with $\mathfrak{D}$ the subspace of $\mathfrak{P}$ consisting of all functions in $\mathfrak{P}$ with compact support. For the topology $\mathfrak{P}$ and $\mathfrak{D}$ see the Schwartz's book ([4]). And we denote $\varphi(x+h)$ with $\varphi_h(x)$ as a function of x. The purpose of this paper is to prove the following

Theorem. *Let T be a continuous linear mapping from $\mathfrak{P}$ to itself which satisfies the following conditions*:

I) $T^2\varphi(x)=\varphi(-x)$,

II) $T(\varphi*\psi)=T\varphi\cdot T\psi$.

Then $T\varphi(x)$ must be equal to $E\varphi(x)$ or $E\varphi(-x)$, where $E\varphi(x)$ is the Fourier transform $\int_{-\infty}^{\infty} e^{2\pi ixt}\varphi(t)\,dt$ *of* $\varphi(x)$.

2. First we shall prove a few lemmas.

Lemma 1. *Let φ, ψ be elements of $\mathfrak{D}$ and the support of φ be contained in $[a,b]$. If we put*

$$f_n(x)=\frac{b-a}{n}\sum_{j=1}^{n}\varphi(x-h_j)\psi(h_j)$$

for every natural number n, where $h_j=a+\frac{(b-a)j}{n}$, then the series $f_1(x), f_2(x), \cdots$ converges to $\varphi*\psi$ in $\mathfrak{D}$ and, a fortiori, *in $\mathfrak{P}$.*

We omit the proof of this lemma because it is very easy.

Lemma 2. *There is a continuous function $r(x)$ on the real number field such that*

$$T\varphi_h(x)=\exp(2\pi ihr(x))T\varphi(x)$$

for every function φ in $\mathfrak{P}$ and every couple of real numbers h and x.

Proof. For any given x there exists an element ψ of $\mathfrak{P}$ such that $T\psi(x)\neq 0$ by Condition I. Let us denote $\frac{T\psi_h(x)}{T\psi(x)}$ with $u(h,x)$ or $u(h)$. Because

$$(\varphi*\psi)_h=\varphi_h*\psi=\varphi*\psi_h$$

we get

$$T\varphi_h(x)T\psi(x)=T\varphi(x)T\psi_h(x)$$

by Condition II. Therefore

$$T\varphi_h(x)=T\varphi(x)u(h)$$

for every φ in $\mathfrak{P}$. From this we can claim $u(h)\neq 0$, because there exists an element φ of $\mathfrak{P}$ such that $T\varphi_h(x)\neq 0$. Also we see that if

$T\varphi(x)\neq 0$ then $T\varphi_h(x)\neq 0$ for all h. And by the fact $\psi_{h+k}=(\psi_h)_k$ we obtain

$$T\psi_{h+k}(x)=T\psi(x)u(h+k)$$
$$=T\psi_h(x)u(k)$$
$$=T\psi(x)u(h)u(k)$$

and
$$u(h+k)=u(h)u(k).$$

Because ψ_h is continuous as a functional of h, $T\psi_h$ is continuous with respect to h and $u(h)$ is also continuous in h. So we can write $u(h,x)$ as $\exp(2\pi ihr(x))$ with some (complex) number $r(x)$. Moreover $r(x)$ is continuous, for $T\psi_1(x)$ and $T\psi(x)$ are continuous in x.

Lemma 3. *There is a real number α such that $r(x)=\alpha x$ for all x.*

Proof. By the hypotheses of the theorem we get

$$T(T\varphi * T\psi)(-x)=T^2\varphi(-x)\cdot T^2\psi(-x)$$
$$=\varphi(x)\psi(x).$$

Applying T to the first and third terms in the above equation we obtain

$$T\varphi * T\psi = T(\varphi\cdot\psi)$$

by Condition I. If we substitute φ_h and ψ_h into this formula we have

$$T\varphi_h * T\psi_h = T(\varphi_h\cdot\psi_h)=T((\varphi\psi)_h),$$

or

$$\int_{-\infty}^{\infty}\exp(2\pi ihr(x-t))T\varphi(x-t)\exp(2\pi ihr(t))T\psi(t)\,dt$$
$$=\exp(2\pi ihr(x))T(\varphi\psi)(x)$$

and

$$\int_{-\infty}^{\infty}\exp(2\pi ih(r(x-t)+r(t)-r(x)))T\varphi(x-t)T\psi(t)\,dt$$
$$=\int_{-\infty}^{\infty}T\varphi(x-t)T\psi(t)\,dt$$

for every φ and ψ in $\mathfrak{P}$. Because the set of every $T\varphi(x-t)T\psi(t)$ with φ and ψ in $\mathfrak{P}$ as a function of t is dense in $\mathfrak{P}$, we get

$$\exp(2\pi ih(r(x-t)+r(t)-r(x)))=1.$$

Therefore
$$r(x-t)=r(x)-r(t)$$

and so there is a number α such that

$$r(x)=\alpha x.$$

Now we shall prove α is a real number. Let $\alpha=\beta+\gamma i$ where β and γ are real numbers with $\gamma\neq 0$, say $\gamma>0$. We take such a function φ in $\mathfrak{P}$ that the support of $T\varphi$ is contained in $[1,2]$. Then the support of $T\varphi_h(x)=\exp(2\pi i\alpha hx)T\varphi(x)$ is also contained in $[1,2]$ and

$$\left|\frac{d^n}{dx^n}T\varphi_h(x)\right|\leq\sum_{m=0}^{n}\binom{n}{m}|2\pi\alpha h|^{n-m}\left|\frac{d^mT\varphi(x)}{dx^m}\right|e^{-2\pi\gamma hx}$$

in $[1,2]$. Therefore $T\varphi_h$ converges to 0 in $\mathfrak{P}$ if h tends to ∞ and $\varphi_h(x)=T\ T\varphi_h(-x)$ converges to 0 in $\mathfrak{P}$ by the continuity of T. But this is impossible. Q.E.D.

3. By Lemmas 2 and 3 we have

$$T\varphi_h(x)=\exp(2\pi i\alpha hx)T\varphi(x) \quad \text{for every } \varphi \text{ in } \mathfrak{P}.$$

Now we consider the functions φ and ψ in Lemma 1 and shall use the notations in the same lemma. Then we have

$$Tf_n(x)=\frac{b-a}{n}\sum_{j=1}^{n} T\varphi_{-h_j}(x)\psi(h_j)=\frac{b-a}{n}\sum_{j=1}^{n}\exp(-2\pi i\alpha h_j x)\psi(h_j)T\varphi(x).$$

And by Lemma 1 and the continuity of T we get

$$T(\varphi*\psi)(x)=\int_{-\infty}^{\infty}\exp(-2\pi i\alpha xh)\psi(h)dh\cdot T\varphi(x).$$

By Condition II we obtain from this formula

$$T\psi(x)=E\psi(-\alpha x) \quad \text{for all } \psi \text{ in } \mathfrak{D}.$$

But this equation is valid for any function in $\mathfrak{P}$ because $\mathfrak{D}$ is dense in $\mathfrak{P}$. And α is different from 0. Then,

$$\psi(-x)=T^2\psi(x)=\frac{1}{|-\alpha|}EE\psi\left(\frac{1}{-\alpha}(-\alpha x)\right)$$

$$=\frac{1}{|\alpha|}E^2\psi(x)=\frac{1}{|\alpha|}\psi(-x).$$

So we get $\alpha=\pm 1$ and

$$T\psi(x)=E\psi(x) \quad \text{or} \quad E\psi(-x).$$

Thus we have completed the proof of the theorem.

References

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