32. *Further Results in Lebesgue Geometry of Curves*

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1. Proof of a theorem. As heretofore we shall be concerned with curves situated in a Euclidean space $\boldsymbol{R}^m$ of dimension $m \geqq 2$. Sets, by themselves, will always mean sets of real numbers unless specified to the contrary. To prove the theorem stated at the end of [4], we shall begin with a lemma in which the points of $\boldsymbol{R}^m$ will be called vectors for convenience.

LEMMA. (i) *We have* $(x \diamond y)|x| < 4|x-y|$ *for every distinct pair of nonvanishing vectors* x *and* y. (ii) *Given a positive number* $\varepsilon \leqq 1/2$ *and four vectors* p, q, p', q' *such that* $p \neq 0, q \neq 0$, *and* $p \diamond q \neq 0$, *write for short* $\theta = (p \diamond q)/4$ *and suppose that*

$$|p'-p| \leqq \varepsilon\theta|p|, \quad |q'-q| \leqq \varepsilon\theta|q|.$$

Then the two vectors $p-q$ *and* $p'-q'$ *are nonvanishing and the angle between them is less than* 8ε.

PROOF. *re* (i): The identity $|x-y|^2 = |x|^2 + |y|^2 - 2|x| \cdot |y| \cos\alpha$, where $\alpha = x \diamond y$, implies that if $\alpha > \pi/2$, then $4|x-y| > 4|x| > \alpha|x|$. On the other hand we always have $|x-y| \geqq |x| \sin\alpha$ on account of the identity $|x-y|^2 - (|x|\sin\alpha)^2 = (|x|\cos\alpha - |y|)^2$. When $\alpha \leqq \pi/2$, we therefore find, in view of the well-known inequality $\pi \sin\alpha \geqq 2\alpha$, that $\alpha|x| \leqq 2|x|\sin\alpha \leqq 2|x-y|$. This establishes (i).

re (ii): Write $w = p-q$ and $w' = p'-q'$, so that $w \neq 0$ since $p \diamond q \neq 0$. Part (i) proved already implies $\theta|p| < |w|$ and $\theta|q| < |w|$. Hence

$$|p'-p| + |q'-q| \leqq \varepsilon\theta|p| + \varepsilon\theta|q| < 2\varepsilon|w|.$$

This, united with the evident relation $|w| \leqq |w'| + |p'-p| + |q'-q|$, gives $|w'| > (1-2\varepsilon)|w| \geqq 0$, so that w' cannot vanish. Putting now for brevity $\lambda = (w \diamond w')/4$ and using (i) again, we find further

$$\lambda|w| \leqq |w-w'| \leqq |p'-p| + |q'-q| < 2\varepsilon|w|.$$

Since $w \neq 0$, it follows that $\lambda < 2\varepsilon$, Q. E. D.

THEOREM. *A light curve* φ *is spherically representable on both sides provided that it is locally straightenable.*

PROOF. We can associate with each point $a \in \boldsymbol{R}$ a positive number δ (depending on a) such that $\varphi(t) \neq \varphi(a)$ whenever $a < t \leqq a+\delta$. For otherwise there would exist a strictly decreasing sequence of points $t_1 > t_2 > \cdots$ tending to a and such that $\varphi(t_n) = \varphi(a)$ for each $n = 1, 2, \cdots$. Consider now the interval $K_n = [t_{n+1}, t_n]$ for each n. Then the curve φ, which is light by hypothesis, could not be constant on K_n, so that $\Omega(\varphi; K_n) \geqq \pi$ on account of [1]§60. In view of superadditivity of bend (see [1]§31) this contradicts the local straightenableness of φ.

This being premised, let us suppose, in order to prove the lemma, that φ has no right-hand tangent direction ([1]§77) at a fixed point c of $\boldsymbol{R}$. As we see at once, φ then possesses at c at least two right-hand derived directions ([1]§73), say α and β. This implies that in each open interval with left-hand extremity c there are three points $c_1<c_2<c_3$ such that the vectors p', q', r' defined by

$$p'=\varphi(c_1)-\varphi(c), \quad q'=\varphi(c_2)-\varphi(c), \quad r'=\varphi(c_3)-\varphi(c)$$

are all nonvanishing and furthermore the angles $p'\diamond\alpha, q'\diamond\beta, r'\diamond\alpha$ are all less than $8\varepsilon^2$, where ε is short for $(\alpha\diamond\beta)/32$. If we now write $p=|p'|\alpha$, $q=|q'|\beta$, $r=|r'|\alpha$, it follows at once that

$$4|p'-p|\leqq 4(p'\diamond\alpha)|p|<\varepsilon(\alpha\diamond\beta)|p|.$$

Similarly we get $4|q'-q|<\varepsilon(\alpha\diamond\beta)|q|$ and $4|r'-r|<\varepsilon(\alpha\diamond\beta)|r|$. But here $\alpha\diamond\beta=p\diamond q=q\diamond r$, and so we deduce from the above lemma that

$$p'\neq q'\neq r', \quad (q-p)\diamond(q'-p')<8\varepsilon, \quad (r-q)\diamond(r'-q')<8\varepsilon.$$

The latter appraisals, conjointly with the triangular inequality for angles (see [1]§22), readily leads to

$$(1) \qquad (q'-p')\diamond(r'-q')>(q-p)\diamond(r-q)-16\varepsilon.$$

Let us now estimate the angle $\lambda=(q-p)\diamond(r-q)$. If $p=r$, then obviously $\lambda=\pi\geqq\alpha\diamond\beta$. Suppose therefore that $p\neq r$. Then the triangular equality of [1]§23 requires that

$$(2) \qquad \lambda=(q-p)\diamond(r-p)+(q-r)\diamond(p-r).$$

We distinguish two cases according as $|p'|<|r'|$ or $|p'|>|r'|$. In the first case we clearly have $p\diamond(r-p)=0$, so that

$$\lambda\geqq p\diamond(q-p)=p\diamond q+(q-p)\diamond q\geqq p\diamond q=\alpha\diamond\beta,$$

where we have applied once more the triangular equality. But the same result $\lambda\geqq\alpha\diamond\beta$ must hold in the second case also, since the right-hand side of (2) is symmetric with respect to the letters p and r. In virtue of (1) it follows now at once that

$$\Omega(\varphi;\{c_1, c_2, c_3\})=(q'-p')\diamond(r'-q')>(\alpha\diamond\beta)/2.$$

This being so, consider the closed interval $I_0=[c, c+1]$. The last inequality enables us to choose in I_0 a disjoint infinite sequence Δ of closed intervals such that $\Omega(\varphi; I)>(\alpha\diamond\beta)/2$ for every interval I in Δ. Then $\Omega(\varphi; I_0)\geqq\Omega(\varphi;\Delta)=+\infty$ by superadditivity of bend. This contradicts local straightenableness of φ. We have thus proved that φ is spherically representable on the right. By symmetry φ must be so on the left too, and the theorem is established.

REMARK. The above theorem completes the proposition of [1]§80. As for the method of proof of our theorem, we may observe the following. Let U be the union of all the open intervals on which φ is rectifiable. Arguing as in the final paragraph of [3] we find that $\boldsymbol{R}-U$ is a countable set. Consider any point t of U. If φ is continuous on the right at t, i.e. if $\varphi(t+)=\varphi(t)$, it follows from part (ii) of the theorem of [4]§3 that φ has at t a right-hand tangent direction. If on the other hand $\varphi(t+)\neq\varphi(t)$, the direction of $\varphi(t+)-\varphi(t)$ is clearly the right-hand tangent direction of φ at t. Similarly we see that φ has at t a left-hand tangent direction. The same argument fails, however, when t does not belong to the set U. For then $\varphi(t+)$ does not necessarily exist, neither does $\varphi(t-)$.

2. Hausdorff measure-bend of a curve. We defined in [2]§5 a set-function $\omega(X)$ for all subsets X of the space $\boldsymbol{R}^m$. With the help of ω we shall now introduce a geometric quantity called Hausdorff measure-bend of a curve. Given a curve φ and a set E of real numbers, let ε be a positive number and express E arbitrarily as the join of a sequence $\varDelta$ of sets with diameters less than ε. Noting that φ is situated in $\boldsymbol{R}^m$, we write for short $\varPhi(M)=\omega(\varphi[M])$ for every linear set M and understand by $\varPi_\varepsilon(\varphi; E)$ the infimum of $\varPhi(\varDelta)$ for all sequences $\varDelta$ of the above description. As $\varepsilon\to 0$, this infimum plainly tends in a non-decreasing manner to a limit, which will be termed *Hausdorff measure-bend* of the curve φ over the set E and denoted by $\varPi(\varphi; E)$. We find easily that (i) if E is covered by a sequence $\varTheta$ of linear sets, then $\varPi_\varepsilon(\varphi; E)\leqq\varPi_\varepsilon(\varphi; \varTheta)$; and that (ii) if M_1 and M_2 are a pair of nonvoid linear sets with distance exceeding ε, then $\varPi_\varepsilon(\varphi; M_1\smile M_2)=\varPi_\varepsilon(\varphi; M_1)+\varPi_\varepsilon(\varphi; M_2)$. It follows at once from these two relations that *the quantity $\varPi(\varphi; E)$, qua function of E, is an outer Carathéodory measure which vanishes for countable sets.*

The following inequality between the Hausdorff and reduced measure-bends will sometimes be useful to us hereafter.

THEOREM. *Given φ and E as above, we have $\varPi(\varphi; E)\leqq\varUpsilon(\varphi; E)$.*

PROOF. Let $\varDelta$ and $\varPhi$ retain the same meanings as above, $\varepsilon>0$ being fixed. We find by the lemma of [2]§4 that the reduced measure-bend $\varUpsilon(\varphi; E)$ is the infimum, for all $\varDelta$, of the sum $\varOmega(\varphi; \varDelta)$. On the other hand $\varPhi(M)\leqq\varOmega(\varphi; M)$ for any linear set M, as may easily be deduced from the definition ([2]§5) of the function $\omega(X)$. Hence $\varPhi(\varDelta)\leqq\varOmega(\varphi; \varDelta)$ for each $\varDelta$, and it follows that

$$\varPi_\varepsilon(\varphi; E)=\inf \varPhi(\varDelta)\leqq\inf \varOmega(\varphi; \varDelta)=\varUpsilon(\varphi; E).$$

Making $\varepsilon\to 0$, we get $\varPi(\varphi; E)=\lim \varPi_\varepsilon(\varphi; E)\leqq\varUpsilon(\varphi; E)$, and the proof is complete.

REMARK. It is easy to construct a continuous plane curve $\varphi_0(t)$ such that $\varPi(\varphi_0; I)\neq\varUpsilon(\varphi_0; I)$ for every closed interval I. For this purpose consider a real-valued continuous function $F_0(t)$ which is defined on $\boldsymbol{R}$ and nowhere differentiable. Then F_0 cannot be VBG on any closed interval, as it follows immediately from a remark (in small print) on p. 234 of Saks [5]. We define now $\varphi_0(t)=\langle F_0(t), 0\rangle$ for every t and readily see that $\omega(\varphi[M])=0$ for every linear set M. Consequently, by definition, the Hausdorff measure-bend of φ_0 vanishes identically. We proceed to prove that, on the other hand, $\Upsilon(\varphi_0; I)\geqq\pi$ for each closed interval I. Let us express I in any manner as the join of a sequence Δ of its subsets. It is clearly sufficient to show that there exists, among the sets composing Δ, one at least on which the curve φ_0 has bend $\geqq\pi$. Suppose on the contrary that $\Omega(\varphi_0; E)<\pi$ for each set E in Δ. The function F_0 must then be monotone (non-decreasing or non-increasing) over each E; for otherwise we could choose in E a triple of points $t_1<t_2<t_3$ so as fulfil either

$$F_0(t_1)<F_0(t_2)>F_0(t_3) \text{ or } F_0(t_1)>F_0(t_2)<F_0(t_3),$$

and it ensues directly that $\Omega(\varphi_0; E)\geqq\Omega(\varphi_0; \{t_1, t_2, t_3\})=\pi$, which is incompatible with our assumption. Since F_0 is bounded on I as a continuous function, monotonity of F_0 on E implies that it is VB on E. This being true for each E in Δ, we conclude that F_0 is VBG on I, contrary to what has already been said above. Thus φ_0 has the announced property $\Upsilon(\varphi_0; I)\geqq\pi$. By the way, it follows further at once from this that $\Upsilon(\varphi_0; J)=+\infty$ for every interval J (closed or not).

3. **A relation between bend and the function ω.** The bend of a curve φ over a set E is sometimes expressible in the form $\Phi(E)$, where Φ has the same meaning as in §2. To obtain a sufficient condition for this to take place, we prove first the following

LEMMA. *Given in $\boldsymbol{R}^m$ a finite sequence $\Delta=\langle x_0, x_1,\cdots, x_n\rangle$ of points, where $n\geqq 2$, suppose that $p_i=x_i-x_{i-1}\neq 0$ for every $i=1,\cdots,n$ and that $\alpha=p_1\diamond p_2+\cdots+p_{n-1}\diamond p_n\leqq\pi/2$. Then Δ is distinct and we have $\omega(X)=\alpha$ for the set $X=\{x_0, x_1,\cdots, x_n\}$.*

PROOF. Throughout the proof the letters j, a, b, c will assume the values $0, 1,\cdots, n$. To see the distinctness of Δ, consider any curve $\xi(t)$ such that $\xi(j)=x_j$ for every j. If there were a pair of values of a, b such that $a<b$ and $x_a=x_b$, we should find at once, taking into account the evident relation $b\geqq a+2$, that

$$\alpha=\Omega(\xi; \{0, 1,\cdots, n\})\geqq\Omega(\xi; \{a, a+1, b\})=\pi,$$

which is a contradiction. This proves Δ distinct.

A distinct triple $\langle x_a, x_b, x_c\rangle$ of points of X will for the moment be termed *compatible* with the sequence Δ iff either $a<b<c$ or $c<b<a$. Consider now any permutation of Δ, say $\Delta'=\langle y_0, y_1,\cdots, y_n\rangle$, and choose a curve $\eta(t)$ such that $\eta(j)=y_j$ for every j. The equality $\omega(X)=\alpha$ will plainly follow if we show that $\beta=\Omega(\eta; \{0, 1,\cdots, n\})\geqq\alpha$. For this purpose let us assume in the first place that the triple $\langle y_0, y_1, y_2\rangle$ is incompatible with Δ. Then, writing $y_0=x_a, y_1=x_b$, and $y_2=x_c$, we find that the indices a, b, c must satisfy one of the following four relations:

$$a<c<b,\quad b<a<c,\quad b<c<a,\quad c<a<b.$$

If $a<c<b$ holds, consider the obvious relation

$$\Omega(\eta; \{0, 1, 2\}) = (x_b - x_a) \diamond (x_c - x_b) = \pi - (x_b - x_a) \diamond (x_b - x_c).$$

Here we have the inequality $(x_b - x_a) \diamond (x_b - x_c) \leqq (x_c - x_a) \diamond (x_b - x_c)$ on account of the triangular equality of [1]§23. But the last angle, being equal to $\Omega(\xi; \{a, c, b\})$, cannot exceed α. It follows that

$$\beta \geqq \Omega(\eta; \{0, 1, 2\}) \geqq \pi - \alpha \geqq \alpha.$$

Of course the same result $\beta \geqq \alpha$ may be deduced similarly in each of the remaining three cases $b < a < c$, etc.

We thus have $\beta \geqq \alpha$ whenever $\langle y_0, y_1, y_2 \rangle$ is incompatible with Δ. But we can clearly replace, in this statement, the triple $\langle y_0, y_1, y_2 \rangle$ by any triple $\Theta_j = \langle y_j, y_{j+1}, y_{j+2} \rangle$, where $j \leqq n-2$. Accordingly $\beta \geqq \alpha$ must hold whenever there exists a Θ_j which is incompatible with Δ. It only remains to examine the case in which every Θ_j is compatible with Δ. But evidently $\Delta' = \langle y_0, y_1, \cdots, y_n \rangle$ then coincides either with Δ or with Δ reversed, i.e. the sequence $\langle x_n, x_{n-1}, \cdots, x_0 \rangle$; so that β coincides with α. This establishes the assertion.

THEOREM. *Given a curve φ and a set E, suppose that $\Omega(\varphi; E) \leqq \pi/2$ and write as before $\Phi(M) = \omega(\varphi[M])$ for each set M. Then we have $\Phi(E) = \Omega(\varphi; E)$.*

PROOF. It being obvious that $\Phi(E) \leqq \Omega(\varphi; E)$, we need only examine the converse inequality. Supposing $\Omega(\varphi; E) > 0$ as we may, consider any finite set $S \subset E$ for which $\Omega(\varphi; S) > 0$. Since $\Omega(\varphi; E)$ is the supremum of $\Omega(\varphi; S)$ for all such S, it is enough to prove $\Omega(\varphi; S) \leqq \Phi(E)$ for each S. We can plainly choose in S a finite sequence of points $t_0 < t_1 < \cdots < t_n$ $(n \geqq 2)$ in such a way that $\varphi(t_{i-1}) \neq \varphi(t_i)$ for every $i = 1, \cdots, n$ and further that $\Omega(\varphi; T) = \Omega(\varphi; S)$, where we write for short $T = \{t_0, t_1, \cdots, t_n\}$. But $\Omega(\varphi; S) \leqq \Omega(\varphi; E) \leqq \pi/2$ by hypothesis, and hence our lemma requires that $\Omega(\varphi; T) = \Phi(T)$. Consequently we have $\Omega(\varphi; S) = \Phi(T) \leqq \Phi(E)$, which completes the proof.

4. Countable and Borel straightenableness of a curve. We shall call a curve φ *countably straightenable* on a set E iff E admits an expression as the union of a sequence (countable, needless to say) of sets on each of which φ is straightenable. When such a sequence can especially be so chosen that each set X composing it has the form $X = BE$, where B is a suitable Borel set (depending on X), φ will be termed *Borel-straightenable* (or *B-straightenable*) on E. It may be shown that each of the following three conditions is sufficient for φ to be Borel-straightenable on E: (i) *φ is continuous on E and countably straightenable on E*; (ii) *φ is locally straightenable*; (iii) *$\Omega_*(\varphi; \{t\}) < +\infty$ whenever $t \in E$.* Plainly condition (ii) implies condition (iii).

As we may observe, condition (i) is the analogue, in bend theory, of a condition of [2]§1 for Borel rectifiability of a curve over a set. But (ii) and (iii) also have their counterparts in length theory, as follows: a curve is Borel-rectifiable on a set E whenever (a) *it is locally rectifiable* or, more generally, (b) *its measure-length is finite for every singletonic subset of* E. (Here the space in which the curve is situated may exceptionally be of arbitrary dimension.) The proof is immediate.

Owing to space limitation the relation between the Hausdorff and reduced measure-bends of a curve has not been fully discussed in the present article. As a typical one of the results to be established in this connection in our forthcoming note we may mention the following (cf. the remark of §2): *If a curve* φ *situated in* $\boldsymbol{R}^m$ *of dimension* $m \geqq 2$ *is Borel-straightenable on a set* E, *it is necessarily Borel-rectifiable on the same set and we further have the equality* $\Pi(\varphi; E) = \Upsilon(\varphi; E)$.

References

[1] Ka. Iseki: On certain properties of parametric curves, Jour. Math. Soc. Japan, **12**, 129–173 (1960).
[2] ——: Some results in Lebesgue geometry of curves, PJA, **37**, 593–598 (1961).
[3] ——: On the reduced measure-bend of curves, PJA, **38**, 37–42 (1962).
[4] ——: Further properties of reduced measure-bend, PJA, **38**, 105–110 (1962).
[5] S. Saks: Theory of the integral, Warszawa-Lwów (1937).