45. On Quasiideals of Regular Ring

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The concept of quasiideal in associative ring was introduced by O. Steinfeld (see [3]). We recall, that a submodule M of an associative ring A is called a quasiideal of A, if and only if the relation

(1) $$AM \cap MA \subseteq M$$

holds. An associative ring is called regular, if for every element $a \in A$ there exists an element $x \in A$ so that $axa = a$. (See J. von Neumann [2].) Recently L. Kovács [1] has proved, that an associative ring A is regular if and only if the relation

(2) $$R \cap L \subseteq RL$$

holds for every left ideal L and for every right ideal R of A.

We prove the following theorem.

Theorem 1. *A submodule M of a regular ring A is a quasiideal of A if and only if M satisfies the condition*

(3) $$MAM \subseteq M.$$

Proof. Let A be a regular ring and let Q be a quasiideal of A. It is easy to see that $QAQ \subseteq AQ$, and $QAQ \subseteq QA$. Hence it follows that

$$QAQ \subseteq QA \cap AQ,$$

and

$$QAQ \subseteq Q,$$

since Q is a quasiideal of A. Therefore the quasiideal Q satisfies (3).

Conversely, suppose that A is a regular ring and M is a submodule of A, satisfying the condition (3). Then since the product AM (MA) is a left (right) ideal of A, it follows from (2) that

(4) $$AM \cap MA \subseteq (MA)(AM).$$

It is evident, that

(5) $$(MA)(AM) \subseteq MAM,$$

and thus from (4), (5), and (3) it follows that

$$AM \cap MA \subseteq M,$$

that is the submodule M is a quasiideal of A. Theorem 1 is proved.

Theorem 2. *Let Q_1, Q_2 be quasiideals of a regular ring A. Then the product Q_1Q_2 is likewise a quasiideal of A.*

Proof. We prove that the product Q_1Q_2 satisfies the condition (3). Since $AQ_1 \subseteq A$, we have

$$(Q_1Q_2)A(Q_1Q_2) \subseteq Q_1 \cdot Q_2AQ_2 \subseteq Q_1Q_2,$$

i.e. the product Q_1Q_2 is a quasiideal of A.

Crorllary. *The set of all quasiiaeals in a regular ring is a semigroup.*

Remark. If a submodule M of an arbitrary ring A, satisfying the condition (3) we call (1, 1)-ideal of A, then maybe to show that the product of two quasiideals of an arbitrary associative ring A is an (1, 1)-ideal of A.

References

[1] L. Kovács: A note on regular rings, Publ. Math. Debrecen, **4**, 465–468 (1956).
[2] J. von Neumann: On regular rings, Proc. Nat. Acad. Sci. U.S.A., **22**, 707–713 (1936).
[3] O. Steinfeld: Über die Quasiideale von Ringen, Acta Sci. Math. (Szeged), **17**, 170–180 (1956).