44. On Complex Dirichlet Principle

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1. Let R, R' be a couple of compact Riemann surfaces with the same positive genus. Consider a conformal metric $\eta = ds_{\mathfrak{q}}^2 = \rho(w)\,|dw|^2$ introduced on R'; $\rho(w)\,|dw|^2$ remains invariant under any conformal transformations of the local parameter w attached to the point $\mathfrak{q} \in R'$ in question; $\rho(w)$ shall be positive and continuous in w.

Given a smooth homeomorphism f, mapping R onto R', we set

$$ds_{\mathfrak{q}}^2 = E_f dx^2 + 2F_f dxdy + G_f dy^2$$
$$= (E_f + G_f)\,|dz|^2/2 + \mathrm{Re}\,\{(E_f - G_f - 2iF_f)dz^2\}/2,$$

where $z = x + iy$ is a local parameter near the point $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Then

$$I[f] = \frac{1}{2}\iint_R (E_f + G_f)dxdy$$

is regarded as a functional in f for the fixed η. In the brief but perspicacious paper due to Gerstenhaber-Rauch [2], one will find the following

DEFINITION. *f is called harmonic relative to η, when the quadratic differential $(E_f - G_f - 2iF_f)dz^2$ is holomorphic on R;*[1)]
and

THEOREM. *If $I[f] \leq I[g]$ holds for any topological mapping g from R to R' homotopic to f, f is harmonic relative to η.*

The content is, however, extremely heuristic and their reasoning involves somewhat essential gaps unfortunately in the management of the extremum problem: no mention was made of an admissibility condition for concurrence maps. It will be desirable to treat this problem from the purely variational point of view, which has motivated to write the present short note. Roughly speaking, variational problems have two demands which seem to be mutually exclusive in appearance; firstly, the *argument* that extremizes the given functional must be again admitted to concurrence (compactness); secondly, all the arguments in a suitable "neighbourhood" of the extremal must be admitted to concurrence (interiority). Therefore our task is to specify the family of admissible maps for this extremum problem.

The class of quasi-conformal mappings brings hardly any advantages to this purpose; the set of all quasi-conformal maps is not

1) If, locally considered, $w(z) \in C^2$ and η is intrinsic metric, the holomorphy of $E - G - 2iF$ implies $\Delta w = 0$ at every point where the Jacobian $|\partial w/\partial z|^2 - |\partial w/\partial \bar{z}|^2$ does not vanish.

compact, while the set of quasi-conformal maps with uniformly bounded dilatation is incompatible with our second requirement. So we are obliged to abstract only the differentiability conditions, giving up the metrical restrictions, from their characteristic properties, which shall be called (Q)-conditions:

Let $w=w(z)$ be a topological mapping from one plane domain onto another.

(Q. 1); *$w(z)$ is L^2-derivable in the sense of Sobolev-Friedrichs* (cf. Bers [1]).

(Q. 2); *$w(z)$ is absolutely continuous in 2-dimensional sense.*

Though our variational process begins with forming the minimizing sequence as usual, it is not in quite an immediate way that the univalency of the limiting map is guaranteed. It makes the proof inevitably complicated.

2. We shall suppose a compact class of conformal metrics introduced on R; more concretely, we consider a family of some "locally holomorphic" differentials on R, whose absolute values we adopt as line-elements inducing the conformal metrics. The local holomorphy has very little to do with the essential feature of the proof, but is only for the sake of technical convenience.

Let $\bigcup_{j=1}^{k} N_j$ be a system of arbitrarily fixed local parameter neighbourhoods covering R. We also fix a considerably fine triangulation $\sum$ of R: $\sum$ is a collection of a finite number of non-overlapping singular 2-simplices S_j $(j=1,2,\cdots,\kappa)$ which fill up the compact 2-dimensional manifold R and whose boundary 1-simplices are composed of analytic arcs: any point $\mathfrak{p}$ of R belongs at least one of the 2-simplices S_j (i.e., $\mathfrak{p}$ may be interior, on the edges of S_j or coincide with one of the vertices of S_j) $(j=1,2,\cdots,\kappa)$; moreover, each S_j is comprised completely in the interior of at least one of the local parameter neighbourhoods N_j $(j=1,2,\cdots,k)$.

We call a linear differential ω on R *locally holomorphic* there, when it fulfills the following conditions. Denote by $|z|\leq 1$ and $|z'|\leq 1$ be local parameter disks corresponding to arbitrary mutually adjacent singular simplices S_l and S_{l+1} respectively. Let $\mathfrak{p}_0$ be an arbitrary point in the singular 1-simplex $S_l \cap S_{l+1}$ with the local parameters ζ and ζ' relative to each disk considered. $S_l \cap S_{l+1}$ will appear on $|z|=1$ as an arc $\widehat{h_1, h_2}$ and on $|z'|=1$ as $\widehat{h'_1, h'_2}$. Suppose that a point $\mathfrak{p}$ approaches $\mathfrak{p}_0$ non-tangentially from interior of S_j $(j=l, l+1)$. Then $z(\mathfrak{p})$ in $|z|<1$ (resp. $z'(\mathfrak{p})$ in $|z'|<1$) approaches ζ (resp. ζ') non-tangentially. Setting $\omega=\tau(z)dz=\tau'(z')dz'$ in the respective local parameters, we shall assume

(i) $|\tau(z)|$ has a non-tangential limit $|\tau(\zeta)|=\lim|\tau(z(\mathfrak{p}))|$ as $z(\mathfrak{p})\to\zeta(\mathfrak{p}_0)$,

$|\tau'(z')|$ has a non-tangential limit $|\tau'(\zeta')|=\lim|\tau'(z'(\mathfrak{p}))|$ as $z'(\mathfrak{p})\to\zeta'(\mathfrak{p}_0)$ with possible exception for a set of $\mathfrak{p}_0$ of linear measure zero on $S_l\cap S_{l+1}$ $(l=1,2,\cdots,\kappa)$,

(ii) $|\tau(\zeta(\mathfrak{p}_0))|\,|d\zeta(\mathfrak{p}_0)|=|\tau'(\zeta'(\mathfrak{p}_0))|\,|d\zeta'(\mathfrak{p}_0)|$ for a. a. $\mathfrak{p}_0\in\bigcup_{j=1}^{\kappa}\partial S_j$,

(iii) $\sum_{j=1}^{\kappa}\int_{\partial S_j}|\omega|\le M$ with a constant M greater than 1,

(iv) ω is free from zeros in the interior of every simplex S_j $(j=1,2,\cdots,\kappa)$.

Let us denote by Ω the set of all the locally holomorphic differentials $\omega=\tau(z)dz$ which are square-summable over R and normalized by the condition

$$||\omega||^2=(\omega,*\omega)=\iint_R|\tau(z)|^2\,|dz\wedge d\bar{z}|=1.$$

Here, $*\omega$ shall denote the differential conjugate to ω and $|dz\wedge d\bar{z}|$ the absolute value of vector product $dz\wedge d\bar{z}$ of the 2-vectors dz and $d\bar{z}$.

Evidently Ω is not empty, since one can construct a linear differential $\omega=\tau(z)dz$ on R, e.g., by Poisson integral, holomorphic in the interior to each S_j $(j=1,2,\cdots,\kappa)$ such that $|\omega|=|\tau(z)|\,|dz|$ is continuous on the whole R.

LEMMA 1. *The family Ω is normal in each singular simplex of Σ.*

LEMMA 2. *Any differential ω of the family Ω determines the distance between two points of R.*

3. Let f be an arbitrary topological mapping from R to R'. Let z be some local parameter defined about a point $\mathfrak{p}_0\in R$, w some local parameter defined about $f(\mathfrak{p}_0)\in R'$ and $w=w(z)$ the function induced by the mapping f. We call $w(z)$ a *local parametric expression* of f in the neighbourhood of $\mathfrak{p}_0$. Suppose now, an arbitrary local parametric expression of f possesses the property (Q) in a neighbourhood of a point $\mathfrak{p}_0\in R$. Then, its all local parametric expressions have the property (Q) there. We may say simply that f has the property (Q) at $\mathfrak{p}_0$, if a local parametric expression of f has this property in a suitable neighbourhood of $\mathfrak{p}_0$.

Let $\mathfrak{F}_\omega$ be the family of mappings f satisfying the following five conditions, which we shall admit as the concurrence mappings to our first extremum problem:

I) *f is a sense-preserving topological mapping from R to R'.*

II) *f belongs to the given homotopy class A.*

III) *The inverse map f^{-1}, as well as f, possesses the property (Q) at every point of the surfaces.*

The homotopy class A contains at least one quasi-conformal mapping (cf. Teichmüller [3]), whose maximal dilatation we denote by K.

IV) *With a local parametric expression* $w(z)$ *of* f

$$\iint_R \rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)|dz\wedge d\bar{z}|\le \frac{1}{2}\left(K+\frac{1}{K}\right)$$

under the normalization

$$\iint_{R'} \rho(w)\,|dw\wedge d\bar{w}|=1.$$

V) *For a fixed* $\omega\in\Omega$

$$\iint_R \left[D_f(\mathfrak{p})+\frac{1}{D_f(\mathfrak{p})}\right]\omega*\omega\le M\left(K+\frac{1}{K}\right),$$

$D_f(\mathfrak{p})$ *standing for the dilatation of the mapping* f *at the point* $\mathfrak{p}$.[2]

Note that $\mathfrak{F}_\omega$ is non-void.

4. THE FIRST EXTREMUM PROBLEM. *Minimize the Dirichlet integral*

$$I[f]=\frac{1}{4}\iint_R \rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)|dz\wedge d\bar{z}|$$

within the family $\mathfrak{F}_\omega$.

In solving this, attention should be concentrated on the compactness consideration.

LEMMA 3. $\mathfrak{F}_\omega$ *is equicontinuous.*

The proof goes by making use of a slight modification of the well-known length-area principle.

THEOREM 1. *The family* $\mathfrak{F}_\omega$ *is normal on* R.

This theorem enables us to define a substantially meaningful minimizing sequence; i.e., it contains at least one subsequence $\{f_n\}_{n=1}^\infty$ which converges uniformly on R, such that

$$0<I_0=\inf_{\mathfrak{F}_\omega} I[f]=\lim_{n\to\infty} I[f_n].$$

Thus we get a limiting map, which shall be denoted by f_ω, since it depends on ω.

PROPOSITION 1. $f_\omega(\mathfrak{p})$ *is a topological mapping from* R *to* R'.

The condition (V) assures the schlichtness of f_ω.

PROPOSITION 2. *Both* f_ω *and* f_ω^{-1} *possess the property* (*Q*).

PROPOSITION 3. f_ω *belongs to the homotopy class* A.

PROPOSITION 4. f_ω *preserves the orientation.*

PROPOSITION 5.

$$\iint_R \left[D_{f_\omega}(\mathfrak{p})+\frac{1}{D_{f_\omega}(\mathfrak{p})}\right]\omega*\omega\le M\left(K+\frac{1}{K}\right).$$

Propositions 1–5 show that $\mathfrak{F}_\omega$ is compact. But hitherto we have not needed the minimizing property of $\{f_n\}_{n=1}^\infty$. Now, taking into consideration the lower semicontinuity of the functional $I[f]$, we have

2) The integrals in these statements are independent of the choice of local parametric expressions.

$$I[f_\omega] \leq \lim_{n\to\infty} I[f_n] = I_0,$$

while in view of compactness of $\mathfrak{F}_\omega$,

$$I_0 \leq I[f_\omega].$$

Therefore, f_ω solves the extremum problem raised in this section.

5. THE SECOND EXTREMUM PROBLEM. Since, to every $\mathfrak{F}_\omega$ there corresponds at least one extremal f_ω, let us now ask how the situation will be, when ω varies over Ω. Put $\mathfrak{F}_\Omega = \{\mathfrak{F}_\omega, \omega \in \Omega\}$. We are going to settle the problem:

Minimize the functional

$$I[f_\omega] = \frac{1}{4}\iint_R \rho(w_\omega(z))\left(\left|\frac{\partial w_\omega}{\partial z}\right|^2 + \left|\frac{\partial w_\omega}{\partial \bar{z}}\right|^2\right)|dz \wedge d\bar{z}|$$

within the family $\mathfrak{F}_\Omega$, $w_\omega(z)$ being a local parametric expression of f_ω.

By definition, there exists a minimizing sequence $\{f_{\omega_n}\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} I[f_{\omega_n}] = \inf_{\mathfrak{F}_\Omega} I[f_\omega].$$

For a moment, we must be interested again in the compactness of the families.

LEMMA 1′. *$\mathfrak{F}_\Omega$ is equicontinuous.*

THEOREM 1′. *The family $\mathfrak{F}_\Omega$ is normal on R.*

Hence the minimizing sequence $\{f_{\omega_n}\}_{n=1}^\infty$ contains at least one subsequence uniformly convergent on R. We denote its limiting map by $g(\mathfrak{p})$.

LEMMA 4. *The family Ω is compact on R.*

LEMMA 5. *Let $\mathfrak{p}_j$ $(j=1,2)$ be arbitrary points on a singular simplex S_ι and let Γ the class of all rectifiable simple arcs which connects the two points $\mathfrak{p}_j$ on S_ι. Then for any $\gamma \in \Gamma$ and for any $\omega = \tau(z)\,dz \in \Omega$, the quantity*

$$\int_\gamma |\omega| = \int_\gamma |\tau(z)|\,|dz|$$

is bounded away from zero.

PROPOSITION 1′. *$g(\mathfrak{p})$ is a topological mapping from R to R'.*

It follows from Theorem 1′ and Lemma 5.

PROPOSITION 2′. *Both g and g^{-1} possess the property (Q).*

PROPOSITION 3′. *g belongs to the homotopy class A.*

PROPOSITION 4′. *g preserves the orientation.*

The class Ω is compact by Lemma 4, so that the sequence $\{\omega_n\}_{n=1}^\infty$ attatched to the minimizing sequence contains at least one subsequence uniformly convergent in $R - \bigcup_{j=1}^{r} \partial S_j$, the limit of which we denote by $\omega_0 = \tau_0(z)dz$; it belongs to Ω.

PROPOSITION 6. *g belongs to $\mathfrak{F}_{\omega_0}$.*

PROPOSITION 7. *The $\inf_{\mathfrak{F}_\Omega} I[f_\omega]$ is attained by f_{ω_0}.*

We verify that $I[f_{\omega_0}]=I[g]$, since the functional is lower semi-continuous.

6. The solution of the second extremum problem just obtained above, in general, depends on the value of M, since Ω and $\mathfrak{F}_\Omega$ themselves were described in terms of M. Thus our results may be stated as follows: *For every M there exists at least one $\omega_M \in \Omega_M = \Omega$ and $f_M \in \mathfrak{F}_{\Omega_M} = \mathfrak{F}_\Omega$, such that $I[f_M] = \inf\limits_{\mathfrak{F}_{\omega_M}} I[f]$.*

Next we assert

PROPOSITION 8. *For some sufficiently large M, there exists at least one $\tilde{\omega}_M \in \Omega_M$, such that f_M belongs to $\mathfrak{F}_{\tilde{\omega}_M}$ and that*

$$\iint_R \left[D_{f_M}(\mathfrak{p}) + \frac{1}{D_{f_M}(\mathfrak{p})} \right] \tilde{\omega}_M * \tilde{\omega}_M < M\left(K + \frac{1}{K}\right).$$

First of all,

$$\iint_R \left[D_{f_M}(\mathfrak{p}) + \frac{1}{D_{f_M}(\mathfrak{p})} \right] \omega_M * \omega_M \leq M\left(K + \frac{1}{K}\right).$$

Suppose $D_{f_M}(\mathfrak{p}) \equiv \text{const.}$ a.e. on R. Then the equality cannot hold in the above estimate, whence the conclusion is assured by putting $\tilde{\omega}_M = \omega_M$. If, on the contrary, $D_{f_M}(\mathfrak{p}) \not\equiv \text{const.}$ at almost all points of R, one may construct $\tilde{\omega}_M$ from ω_M, utilizing a method which is reminiscent of a "transfer of mass" in the potential theory.

We have found an f_M, which minimizes $I[f]$ in the family $\mathfrak{F}_{\tilde{\omega}_M}$. What is more, f_M is interior to $\mathfrak{F}_{\tilde{\omega}_M}$; f_M is slightly deformable within the family. Since $I[f_M]$ is locally minimal, proceeding just as in the proof of Weyl's lemma, we finally get

THEOREM 2. *There exists at least one topological mapping from R to R' which belongs to the given homotopy class and is harmonic relative to the given conformal metric on R'.*

References

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