# 42. On the Behaviour of Analytic Functions on the Ideal Boundary. III

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(Comm. by K. KUNUGI, M.J.A., May 12, 1962)

*Harmonic measurability and capacity of Borel sets in B.*

**Theorem 3.** *Let R be a Riemann surface with positive boundary. Suppose a topology is given on* $R^*=R+B$. *If* $w_{F_i}(F_j, z)=0$ for $i\neq j$ *for any two closed sets* $F_1$ *and* $F_2$ *in B such that* $\text{dist}(F_1, F_2)>0$. *Then every Borel set in B is harmonically measurable. We call such a topology an H-measurable topology. Especially Stoilow's, Green's, N and K-Martin's topologies are H-measurable.*

Proof. By P.H.3[1] $w(F\cap C\Omega_{1-\varepsilon}, z)=0: \Omega_{1-\varepsilon}=E[z\in R: w(F, z)>1-\varepsilon]$. Hence $w(F, z)=w(F\cap \Omega_{1-\varepsilon}, z)+w(F\cap C\Omega_{1-\varepsilon}, z)=w(F\cap \Omega_{1-\varepsilon}, z)$. Next by $w(F, z)\leqq 1$ $w_{F\cap C\Omega_{1-\varepsilon}}(F, z)\leqq w(F\cap C\Omega_{1-\varepsilon}, z)=0$. Now $w(F, z)\geqq 1-\varepsilon$ on $\Omega_{1-\varepsilon}$. And $w_{F\cap\Omega_{1-\varepsilon}}(F, z)+w_{F\cap C\Omega_{1-\varepsilon}}(F, z)\geqq w_F(F, z)\geqq w_{F\cap\Omega_{1-\varepsilon}}(F\cap\Omega_{1-\varepsilon}, z)$ $\geqq(1-\varepsilon)w(F\cap \Omega_{1-\varepsilon}, z)=(1-\varepsilon)w(F, z)$.

Let $\varepsilon\to 0$. Then $w_F(F, z)=w(F, z)$.

Suppose the topology is $H$-measurable. Then $w_{F_i}(F_j, z)=0: i\neq j: \text{dist}(F_1, F_2)>0$. Whence

$$w_{F_1}(F_1, z)\leqq w_{F_1+F_2}(F_1, z)\leqq w_{F_1}(F_1)+w_{F_2}(F_1, z)=w_{F_1}(F_1, z). \tag{5}$$

Similarly $w_{F_1+F_2}(F_2, z)=w(F_2, z)$. Put $F_{i,n}=E\left[z\in \overline{R}: \text{dist}(z, F_i)\leqq\frac{1}{n}\right]$. Now $w_{F_{i,n}}(F_j, z)=w(F_j, z)$ and $w(F_1+F_2, z)\geqq w(F_i, z)$ on $\partial F_{i,n}$, whence we have

$w_{F_{1,n}}(F_2, z)+w_{F_{2,n}}(F_1, z)+w(F_1+F_2, z)\geqq w(F_2, z)+w(F_1, z)$ on $\partial F_{1,n}+\partial F_{2,n}$. Since ${}_{F_{1,n}+F_{2,n}}(w(F_1, z)+w(F_2, z))$ is the least positive superharmonic function in $R-F_{1,n}-F_{2,n}$ larger than $w(F_1, z)+w(F_2, z)$ on $\partial F_{1,n}+\partial F_{2,n}$, $w_{F_{1,n}}(F_2, z)+w_{F_{2,n}}(F_1, z)+w(F_1+F_2, z)\geqq {}_{F_{1,n}+F_{2,n}}(w(F_1, z)+w(F_2, z))$ in $R-F_{1,n}-F_{2,n}$. Let $n\to\infty$. Then by (5)

$$w(F_1+F_2, z)\geqq w(F_1, z)+w(F_2, z). \tag{6}$$

For any set $A$ we define $w(A, z)=\lim_n w(F_n, z)$, where $F_n\subset A, F_n\uparrow$ and $F_n$ is closed. Then it can be proved by (6) $w(A_1+A_2, z)=w(A_1, z)+w(A_2, z)$ for $\text{dist}(A_1, A_2)>0$. On the other hand, $w(A_1+A_2, z)\leqq w(A_1, z)+w(A_2, z)$ and $w(\sum A_n, z)\leqq\sum w(A_n, z)$ are clear by definition for any sets $A_1, A_2$ and any sequence $\{A_n\}$. Thus $w(A, z)$ is *outer measure of Carathéodry.* For an open set $G$ in $B$, $w(G, z)$ is defined as above. Then

1) Potentials on Riemann surfaces, Journ. Faculty of Science, Hokkaido Univ. (1962).

every Borel set is harmonically measurable, i.e. there exist a sequence of closed sets $F_n\uparrow: F_n\subset A$ and a sequence of open sets $G_n\downarrow: G_n\supset A$ such that

$$\lim_n w(F_n, z)=\lim_n w(G_n, z).$$

1). *For Stoilow's topology.* Let $A_i$ be a closed set such that $\mathrm{dist}(A_1, A_2)>0$. Since $B$ is totally disconnected, we can find a domain $G$ with compact relative boundary such that $\bar{G}\supset A_1$ and $CG\supset A_2$. Let $w(z)$ be the least positive superharmonic function in $G$ such that $w(z)=1$ on $\partial G$. Then there exists a constant $M$ such that $w(z)\leqq MG(z, p)$, where $G(z, p)$ is a Green's function with pole in $CG$. Let $U_{B_n}(z)$ be the least positive superharmonic function such that $U_{B_n}(z)\geqq G(z, p)$ on $B_n$. Then $U_B(z)=\lim_n U_{B_n}(z)=0$, whence we have $w_{A_1}(A_2, z)\leqq MU_B(z)=0$. Thus Stoilow's topology is $H$-measurable.

2). *For Green's topology.* Map the universal covering surface of $R$ onto $|\xi|<1$ by $z=z(\xi)$. Then $z(\xi)$ has angular limits a.e. on $|\xi|=1$. Let $\xi(A_1)$ be the image of $A_1$. Then for any given positive number $\varepsilon$, there exists a closed set $F\subset\xi(A_1)$ and numbers $n$ and $m$ such that $\mathrm{mes}\,(\xi(A_1)-F)<\varepsilon$, the image of $A_{2,n}=E\left[z\in R: \mathrm{dist}(A_2, z)\leqq\frac{1}{n}\right]$ $:\frac{1}{n}<\mathrm{dist}(A_1, A_2)$ does not fall in $D=\bigcup_{e^{i\theta}\in F} E\left[\xi: \arg\left|\frac{\xi-e^{i\theta}}{e^{i\theta}}\right|<\frac{\pi}{4}, |\xi|>1-\frac{1}{m}\right]$. Whence $w(A_2, z)$ has angular limits$=0$ a.e. on $\xi(A_1)$ and $w_{A_1}(A_2, z)=0$.

3). *For N-Martin's topology.* Put $\Omega_{1-\varepsilon}=E[z\in R: \omega(A_1, z)>1-\varepsilon]$ and $\Omega^W_{1-\varepsilon}=E[z\in R: w(A, z, R-R_0)>1-\varepsilon]$. Then by $\omega(A_1, z)\geqq w(A_1, z)$ $\Omega^W_{1-\varepsilon}\subset\Omega_{1-\varepsilon}$. By H.P.4 $w(A_1, z, R-R_0)\leqq w(A_1\cap\Omega^W_{1-\varepsilon}, z, R-R_0)+w(A_1\cap C\Omega^W_{1-\varepsilon}, z, R-R_0)=w(A_1\cap\Omega^W_{1-\varepsilon}, z, R-R_0)\leqq w(\Omega^W_{1-\varepsilon}, z, R-R_0)$. Map the universal covering surface $(R-R_0)^\infty$ of $(R-R_0)$ onto $|\xi|<1$. Then $w(A_1, z, R-R_0)$ has angular limits a.e. on $|\xi|=1$. Let $E^{1-\delta}_\delta$ be the set on which $w(A_1, z, R-R_0)$ has angular limits between $\delta$ and $1-\delta$. Assume $\mathrm{mes}\,E^{1-\delta}_\delta>0$. Then the image of $\Omega^W_{1-\varepsilon}: \varepsilon<\delta$ does not tend along Stolz's path terminating at $E^{1-\delta}_\delta$. Whence $w(A_1, z, R-R_0)=0$ a.e. on $E^{1-\delta}_\delta$. This is a contradiction. Hence $\mathrm{mes}\,E^{1-\delta}_\delta=0$ and $w(\overset{W}{\Omega}{}^{1-\delta}_\delta\cap B, z, R-R_0)=0$: $\overset{W}{\Omega}{}^{1-\delta}_\delta=E[z\in R: \delta<w(A_1, z, R-R_0)<1-\delta]$. Let $A_1$ and $A_2$ be two closed sets in $B$ such that $\mathrm{dist}(A_1, A_2)>0$. Then $w_{A_2}(A_1, z, R-R_0)\leqq w(A_2\cap\Omega^W_{1-\varepsilon}, z, R-R_0)+w(A_2\cap\overset{W}{\Omega}{}^{1-\varepsilon}_\varepsilon, z, R-R)+\varepsilon$. Let $\varepsilon\to 0$. Then

$$w_{A_2}(A_1, z, R-R_0)\leqq w(A_2\cap\Omega^W_{1-\varepsilon}, z, R-R_0)\leqq\omega(A_2\cap\Omega_{1-\varepsilon}, z). \tag{7}$$

Let $A_{2,n}=E\left[z\in\bar{R}: \mathrm{dist}\,(A, z)\leqq\frac{1}{n}\right]$ and $n_2>n_1>0$ such that $A_{2,n_2}\cap A_{1,n_1}$

$=0$. Clearly $\omega(\partial A_{2,n_2}, z) \geqq \omega(A_2, z)$ in $R-R_0-A_{2,n_2}$. Consider $\partial A_{1,n_1}$ as $\partial G$ and $A_{2,n_2}$ as $C(r_1, p)$. Then by (1) we have $w(\Omega^W_{1-\varepsilon} \cap A_2, z, R-R_0) \leqq w(\Omega^*_{1-\varepsilon} \cap A_2, z) \leqq \omega(\Omega^*_{1-\varepsilon} \cap B \cap A_{2,n_2}, z) \downarrow 0$ as $\varepsilon \to 0$, where $\Omega^*_{1-\varepsilon} = E[z \in R: \omega(\partial A_{1,n_1}, z) > 1-\varepsilon]$. Hence by (7) $w_{A_2}(A_1, z, R-R) = 0$. Next we show $w_{A_2}(A_1, z) = 0$. Let $w^*(z)$ be the least positive harmonic function in $R-R_0$ with $w(z)=1$ on $\partial R_0$ and let $w^*_{B_n}(z)$ be the least positive superharmonic function in $R$ such that $w^*_{B_n}(z) \geqq w(z)$ in $B_n = E\left[z \in R: \text{dist}(z, B) \leqq \frac{1}{n}\right]$. Then since $R$ is of positive boundary $\lim_n w^*_{B_n}(z) = 0$. Clearly $w(A_1, z, R-R_0) + w^*(z) \geqq w(A_1, z)$. Hence $0 = w_{A_2}(A_1, z, R-R_0) + w^*_{A_2}(z) \geqq w_{A_2}(A_1, z)$ by $\lim_n w^*_{B_n}(z) \geqq w^*_{A_2}(z)$. Thus $N$-Martin's topology is $H$-measurable.

4). *For K-Martin's topology.* We have by Lemma 7 $w_{A_2}(A_1, z) = 0$ and $K$-Martin's topology is $H$-measurable.

Let $R$ be a Riemann surface with $N$-Martin's topology. Let $F$ be a closed in $\bar{R}-R_0$. Then we defined Cap$(F)$ by $\int_{\partial R_0} \frac{\partial}{\partial n} w(F, z)\, ds$ and $\overset{*}{\text{Cap}}(F)$ by $\frac{1}{\inf_{\mu \in Ca} I(\mu)}$, where $\inf_{\mu \in Ca} I(\mu)$ is the infinimum of Energy Integral of all positive canonical mass distributions of mass unity on $F$ and we showed that there exists a canonical mass distribution $\mu$ on $F$ such that $I(\mu) = \inf_{\mu \in Ca} I(\mu)$, $\omega(F, z) = \int N(z, p)\, d\mu(p)$ and Cap$(F)$ $= \overset{*}{\text{Cap}}(F)$.[2] For any set we define ${}_{in}\text{Cap}(A)$ by $\sup \text{Cap}(F): F$ is closed and $\subset A$. Now since $\mu$ is Borel measurable (with respect to $N$-Martin's topology), we have at once the following

**Proposition.** *Let A be a Borel set. If* ${}_{in}\text{Cap}(A_i) = 0$, *then* ${}_{in}\text{Cap}(\sum A_i) = 0$.

**Lemma a).** *For Fatou's theorem. Let R be a basic surface with positive boundary and with H.S. topology. Let $w=f(z)$ be an analytic function: $z \in R$, $w \in \underline{R}$. Let R be a covering surface with another H.S. topology. Let F be a closed set in B of R and let G be one domain of $f^{-1}(C(r_1, p))$. If $w(F \cap G, z) > 0$, then $w(F \cap G, z, G') > 0$, where $G'$ is one of component of $f^{-1}(C(r_2, p))$ containing $G: r_2 > r_1 > 0$.*

*Proof.* Put $B'_n = f^{-1}(\underline{B}_n)$: $\underline{B}_n = E\left[w \in \underline{R}: \text{dist}(w, \underline{B}) \leqq \frac{1}{n}\right]$ and $CB'_n = R-B_n$. $w(F \cap G \cap B'_n, z) + w(F \cap G \cap CB_n, z) \geqq w(F \cap G, z)$. Hence if $\lim_n w(F \cap G \cap CB'_n, z) = 0$, $w(F \cap G \cap B', z) = \lim_n w(F \cap G \cap B'_n) > 0$. *Case 1.* $\lim_n w(F \cap G \cap CB'_n, z) > 0$. In this case we can find compact circles $\Gamma_1$ and $\Gamma_2$ in $C(r_1, p)$ such that $\Gamma_2 \supset \Gamma_1$ and dist$(\partial\Gamma_1, \partial\Gamma_2) > 0$ and that

2) See 1).

$w(F\cap G\cap G_1, z)=U(z)>0$ where $G_1$ is one component of $f^{-1}(\Gamma_1)$ contained in $G$. Now $U(z)$ is the least positive superharmonic function in $R$ such that $U(z)\geqq 1$ on $F\cap G\cap G_1$. Hence $U(z)\leqq w(\Gamma_1, w)$ on $f^{-1}(\Gamma_1)$. By the compactness of $\partial\Gamma_2$, $\max_{z\in\partial\Gamma_2} w(\Gamma_1, w)<\delta<1$ and $U_{CG_2}(z)\leqq\delta$ on $\partial G_2$ and $U_{CG_2}(z)\leqq\delta$ in $R$, where $G_2$ is a component of $f^{-1}(\Gamma_2)$ containing $G_1$. Hence by $\sup w(F\cap G\cap G_1, z)=1$ (by P.H.2) $w(F\cap G\cap G_1, z)-w_{CG_2}(F\cap G\cap G_1, z)>0$, whence by Theorem 3 of $S$[3] $w(F\cap G\cap G_1, z, G_2)>0$ and $w(F\cap G, z, G')>0$.

*Case 2.* $\lim_n w(F\cap G\cap CB'_n, z)=0$. In this case $w(F\cap G\cap B', z)>0$. Let $G'$ be one component of $f(C(r_2, p))$ containing $G$. Then $w_{CG'}(F\cap B', z)$ is the least positive superharmonic function in $G'$ such that $w_{CG'}(F\cap B', z)\geqq w(F\cap B', z)$ on $\partial G'$ $\{w(\partial C(r_2, p), w)=1$ on $\partial C(r_2, p)$ and $f(\partial G')\subset f^{-1}(\partial C(r_2, p))\}$, hence $w_{CG'}(F\cap B', z)\leqq w(\partial C(r_2, p), w)$, in $C(r_1, p)$ and $f(\Omega^z_{1-\varepsilon}\cap G)\subset(\Omega^W_{1-\varepsilon}\cap C(r_1 p))$, where $\Omega^z_{1-\varepsilon}=E[z\in R: w_{CG'}(F\cap B', z)>1-\varepsilon]$ and $\Omega^W_{1-\varepsilon}=E[w\in R: w(\partial C(r_2, p), w)>1-\varepsilon]$. Now the topology on $\underline{R}$ is H.S. Hence $w(\Omega^z_{1-\varepsilon}\cap G\cap B', z)\leqq w(\Omega^W_{1-\varepsilon}\cap C(r_1, p)\cap B, w)\leqq\frac{1}{1-\varepsilon}w_{B\cap C(r_1, p)}(\partial C(r_2, p), w)\to 0$ as $\varepsilon\to 0$. By P.H.3 $w(F\cap B'\cap C\mho_\varepsilon, z)=0$ $\left(\mho_\varepsilon=E\left[z\in R: w(F\cap B', z)\geqq 1-\frac{\varepsilon}{2}\right]\right)$, whence by $w(F\cap G\cap B', z)>0$ there exists a number $\varepsilon_0$ such that $w(F\cap B'\cap C\Omega^z_{1-\varepsilon_0}\cap\mho_{\varepsilon_0}, z)>0$ and $C\Omega^z_{1-\varepsilon_0}\cap\mho_{\varepsilon_0}\cap R\neq 0$ in which $w(F\cap B'\cap G, z)-w_{CG'}(F\cap B'\cap G, z)\geqq\frac{\varepsilon_0}{2}$. Hence $w(F\cap B'\cap G, z)-w_{CG'}(F\cap B'\cap G, z)>0$ and $w(F\cap B'\cap G, z, G')>0$.

Let $\underline{R}$ be a Riemann surface with null-boundary with H.S. topology and let $R$ be a convering surface with positive boundary. Map the universal covering surface $R^\infty$ of $R$ onto $|\xi|<1$ by $z=z(\zeta)$. If $w=f(z(\zeta))$ has angular limits a.e. on $|\zeta|=1$, we call $f(z)$ a function of $F$-type. On the other hand, the characteristic function $T(z)$ of $f(z)$ can be defined. If $T(z)<\infty$, we call $f(z)$ a function of bounded type. And the characteristic $T(\zeta)$ of $f(z(\zeta))\leqq T(z)$. In this case $f(\zeta)$ has angular limits a.e. on $|\zeta|=1$[4] (using original Fatou's theorem). Suppose $w=f(z)$ is of $F$-type. Then mes $E_{\underline{B}}=0$ by Riesz's theorem, where $E_{\underline{B}}$ is the set on $|\zeta|=1$ on which $f(\zeta)$ has angular limits contained in $\underline{B}$. We shall prove

**Lemma a′).** *Let $R$ be a Riemann surface with null-boundary and let $f(z)$ be a function of $F$-type. Let $F$ be a closed set in $B$ of $R$. If $w(F\cap G, z)>0$, then $w(F\cap G, z, G')>0$, where $G$ is a component of*

3) $S$ means "Singular points of Riemann surfaces", Journ. Faculty of Science, Hokkaido Univ. (1962).

4) Z. Kuramochi: Dirichlet problem on Riemann surfaces. I, Proc. Japan Acad., **30**, 946–950 (1954).

*$f^{-1}(C(r_1, p))$ and $G'$ is one domain of $f^{-1}(C(r_2, p))$ containing $G: r_2 > r_1 > 0$.*

*Proof.* Let $E$ be the set on $|\zeta|=1$ on which $w=f(\zeta)$ has angular limits contained in $\overline{C(r_1, p)}$ (closure of $C(r_1, p)$). Then since $f(\zeta)$ is of $F$-type, $0<w(F\cap G, z)\leqq w(B\cap \overline{f^{-1}(C(r_1,p))}, z)=w(E,\zeta)$, where $w(E,\zeta)$ is the harmonic measure of $E$ with respect to $|\zeta|<1$. Now $w(E,\zeta)=0$ a.e. on $CE$. Let $E_g$ be the set on which a Green's function $G(z, p)$ of $R$ has angular limits$=0$. Then $\text{mes}\, E_g=2\pi$. By $0<w(F\cap G, z)$ and since $f(\zeta)$ is of $F$-type, we can find a positive number $\delta_0$ and a set $E'$ in $E\cap E_g$ such that $\text{mes}\, E'>\delta_0$, $w(F\cap G, z)>\delta_0$ in $G(\delta_0, E')$ and $\text{dist}\,(f(\zeta), C(r_1, p))<\frac{1}{2}(r_2-r_1)$ in $G(\delta_0, E')$, where

$$G(\delta_0, E')=\bigcup_{e^{i\theta}\in E'} E\left[\zeta : \arg\left|\frac{\zeta-e^{i\theta}}{e^{i\theta}}\right|<\frac{\pi}{4},\ 1>|\zeta|>1-\delta_0\right].$$

Now $w(F\cap G, z, G')\geqq w(F\cap G, z)-w(z)$, where $w(z)=\lim_n w_n(z)$ and $w_n(z)$ is a harmonic function in $G\cap R_n$ such that $w_n(z)=w(F\cap G, z)<1$ on $\partial G'$ and $=0$ on $\partial R_n\cap G'$ and $w(F\cap G, z, G')$ is H.M. of $F\cap G$ relative to $G'$. Since the image of $\partial G'$ does not fall in $G(\delta_0, E')$ by $\text{dist}\,(f(\partial G'), C(r_1, p))\geqq r_2-r_1$ and the image of $\partial R_n$ does not tend to $E'$ in $G(\delta_0, E')$ by $\inf_{z\in\partial R_n} G(z, p)>0$, $w_n(z)\leqq w(\zeta)$ and $w(z)\leqq w(\zeta)$, where $w(\zeta)$ is a harmonic function in $G(\delta_0, E')$ such that $w(\zeta)=1$ on $\partial G(\delta_0, E')-E'$ and $=0$ a.e. on $E'$. But $w(F\cap G, z)>\delta_0$ a.e. on $E'$, whence $w(F\cap G, z)-w(\zeta)>\delta_0$ a.e. on $E'$ and $w(F\cap G, z, G')>0$.