# 41. On the Behaviour of Analytic Functions on the Ideal Boundary. II

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In the previous paper[1] with the same title we discussed the theorems of Fatou and Beurling in case where the basic surface is the $w$-Riemann sphere. In the present paper we consider them when the basic surfaces are also abstract Riemann surfaces. Some results contained in this paper is the same one obtained by C. Constantinescu and A. Cornea.[2] All proofs[3] of Fatou's theorem (and that of Constantinescu and Cornea) depend on the following Lebesgue's theorem: "A function of bounded variation has derivatives almost everywhere". We shall prove the Fatou's theorem and Beurling theorem (in extended form) by using only the potentials and the behaviour of the covering surface of an analytic function $w=f(z)$ without the above Lebesgue's theorem. And we shall show that the above two theorems can be proved by the same manner. We denote by $w=f(z): z\in R$ and $w\in \underline{R}$ an analytic function from $R$ into $\underline{R}$.

Let $R_n$ $(n=0,1,2,\cdots)$ be an exhaustion with compact relative boundary $\partial R_n$. We suppose that $R$ is a metric space such that the topology induced by this metric is homeomorphic to the original topology (induced by local parameters) of $R$ when it is restricted in $R$. We have the ideal boundary $B$ of $R$ by the completion of $R$ with respect to the above metric. Then $\overline{R}=R+B$ is closed. In the following distance, closed sets, etc., are ones with respect to the metric on $R$. Put $B_n=E\left[z\in \overline{R}: \mathrm{dist}(z,B)\leqq \frac{1}{n}\right]$. Let $C(r,p)$ be a circle : $C(r,p)=E[z\in \overline{R}: \mathrm{dist}(z,p)<r]$, $p\in \overline{R}$. Suppose that $R$ is a Riemann surface with positive boundary. Put $\Omega_{1-\varepsilon}=E[z\in R: w(\partial C(r_2,p),z)>1-\varepsilon]$. If $\lim_{\varepsilon\to 0} w(\Omega_{1-\varepsilon}\cap C(r_1,p)\cap B,z)=0$ for $r_2>r_1$, we call the topology an H. S. (Harmonically Separative) topology, where $w(\Omega_{1-\varepsilon}\cap C(r_1,p)\cap B,z)$ is H. M. (Harmonic Measure)[4] of $\Omega_{1-\varepsilon}\cap C(r_1,p)\cap B$.

1) On the behaviour of analytic functions on the ideal boundary. I, Proc. Japan Acad., **38**, 150-155 (1962).

2) C. Constantinescu and A. Cornea: Über das Verhalten der analytischen Abbildungen Riemannscher Flächen auf dem idealen Rand von Martin, Nagoya Math. Journ. (1960).

3) For instance S. Lojasiewicz: Une démonstration du théorèm de Fatou, Annales de la société polonaise de mathématique, **22** (1950).

4) Notations and terminologies are to be reffered to "Potentials on Riemann surfaces" and "Singular points of Riemann surfaces", Journ. Faculty of Science, Hokkaido Univ. (1962).

Suppose C. P. (Capacitary Potential) of $C(r_1, p)\cap B$, $\omega(C(r_1, p)\cap B, z) > 0$, where $\omega(C(r_1, p)\cap B, z)=\lim\limits_n \omega_n(z)$ and $\omega_n(z)$ is a harmonic function in $R-R_0-(C(r_1, p)\cap B_n)$ such that $\omega_n(z)=0$ on $\partial R_0$, $\omega_n(z)=1$ on $C(r_1, p)\cap B_n$ and $\omega_n(z)$ has M.D.I. (Minimal Dirichlet Integral). If there exists an increasing sequence of domains $\{V_n\}$ such that $\omega(C(r_1, p)\cap CV_n\cap B, z)\downarrow 0$, $\omega(C(r_1, p)\cap V_n, z)>0$ as $n\to\infty$ and that there exists at least one continuous function $U_n(z)$ in $C(r_2, p)-(C(r_1, p)\cap V_n)$ such that $U_n(z)=1$ on $(V_n\cap C(r_1, p))$, $U_n(z)=0$ on $\partial C(r_2, p)$ and $D(U_n(z))<\infty$ for every[5] $n$, we call such a topology a D.S. (Dirichlet Separative) topology. If $R$ is a Riemann surface with null-boundary, clearly $\omega(C(r_1, p)\cap B, z)=0$.

**Topologies on Riemann surfaces.** 1). *Stoilow's metric.* $R-R_n$ is composed of a finite number of disjoint non-compact surfaces $G_i$. Let $G_n$ $(n=1,2,\cdots)$ be a sequence of non-compact surfaces with compact relative boundary such that $G_n\supset G_{n+1},\cdots$, $\cap G_n=0$. Two sequences $\{G_m\}$ and $\{G'_n\}$ are called equivalent if and only if for any given number $m$ there exists a number $n$ such that $G_m\supset G'_n$ and vice versa. We make correspond an ideal boundary point $p$ (component) to the class of equivalent sequences (which corresponds to neighbourhoods of $p$) and denote the set of all boundary points by $B$. A metric can be introduced on $R+B$. It is clear that $R+B$ and $B$ are *compact* and $B$ is *totally disconnected* and that the topology by this metric is *homeomorphic to the original topology* in $R$.

2). *Martin's topologies.*[6] Let R be a Riemann surface with positive boundary (if $R$ is a surface with null-boundary, consider $R-R_0$ as $R$). Then $K$-Martin's topology can be defined on $R+B$ and $N$-Martin's only on $R-R_0+B$. Above topologies are *homeomorphic to the original topology* in $R-R_0$. We extend them into $R_0$. In this paper we suppose that Martin's topologies are defined on $R+B$.

3). *Green's metric.* Let $R$ be a Riemann surface with positive boundary (if $R$ is a surface with null-boundary, we consider $R-R_0$ instead of $R$). Let $G(z, p)$ be a Green's function. Let $l$ be a curve in $R$. We define the length of $l$ by $\int_l d\,|\,e^{-G(z,p)-ih(z,p)}|$, where $h(z, p)$ is the conjugate function of $G(z, p)$. For two points $p_1$ and $p_2$ of $R$ $\text{dist}(p_1, p_2)$ is defined by the infinimum of the length of all curves connecting $p_1$ with $p_2$ in $R$. Now all boundary points are defined by the completion of $R$ with respect to this metric. It is clear that $R+B$ and $B$ are *closed* but *not always compact.* If this topology is defined only on $R-R_0+B$, we continue this into $R_0$ as Martin's topology.

**Theorem 2.** *Let $R$ be a Riemann surface with positive boundary.*

5) This is equivalent to $D(\omega(V_n\cap C(r_1, p),\ C(r_2, p)))<\infty$.

6) See 4).

*Then Stoilow's, Green's, N and K-Martin's topologies are H.S. topologies. And Stoilow's, Green's, and N-Martin's topologies are D.S. topologies.*

1). *For Stoilow's topology.* Since $B$ is totally disconnected, we can find an open set $G$ with compact and analytic relative boundaries $\partial G$ such that $C(r_1,p)\subset G\subset C(r_2,p)$ and $\partial G\cap B=0$. Clearly $w(\partial C(r_2,p),z)\leqq w(\partial G,z)$ in $G$. Let $G(z,p)$ be a Green's function with pole in $R-C(r_2,p)$. Then $\min_{z\in\partial G} G(z,p)=\delta>0$. Hence $w(\partial G,z)\leqq\frac{G(z,p)}{\delta}$ in $C(r_1,p)$. But $w(\Omega^g_\varepsilon\cap B,z)=0$:[7] $\Omega^g_\varepsilon=E[z\in R: G(z,p)>\varepsilon]$. Hence $w(\Omega_{1-\varepsilon}\cap B\cap C(r_1,p),z)=0$ for $1-\varepsilon>0$ and Stoilow's topology is H.S. (harmonically separative). Suppose $CV_n=0$. As above we can find two domains $G_1$ and $G_2$ with analytic relative boundaries such that $C(r_1,p)\subset G_1\subset G_2\subset C(r_2,p)$ and $\partial G_i\cap B=0$ and $\partial G_1\cap\partial G_2=0$. Now we can easily construct a harmonic function in $G_2-G_1$ such that $U(z)=0$ on $\partial G_2$, $U(z)=1$ on $\partial G_1$ and $D(U(z))<\infty$. Put $U'(z)=0$ in $C(r_2,p)-G_2$, $U'(z)=U(z)$ in $G_2-G_1$ and $U'(z)=1$ in $G_1\supset C(r_1,p)$. Then $U'(z)$ is the function required and Stoilow's topology is D.S.

2). *For Green's topology.* Map the universal covering surface $R^\infty$ of $R$ onto $|\xi|<1$ by $z=z(\xi)$. Then $z(\xi)$ has angular limits with respect to Green's topology[8] a.e. (almost everywhere) on $|\xi|=1$ (Original Fatou's theorem is used for this assertion). Let $B_\xi$ be the image of $B\cap\partial C(r_2,p)$. Then $w(\partial C(r_2,p)\cap B,z)=0$ a.e. on the image of $C(r_1,p)\cap B$ by $\operatorname{dist}(\partial C(r_2,p), C(r_1,p))>0$. Hence we can prove similarly as the proof of Lemma a′) $w(\Omega_{1-\varepsilon}\cap C(r_1,p),z)=0$ for $1-\varepsilon>0$. Thus Green's topology is H.S. Let $G(z,p)$ be the Green's function (used for Green's metric). Put $\zeta(z)=\exp(G(z,p)+ih(z,p))=re^{i\theta}$. We cut $R$ along the trajectries $(h(z,p)=\text{const})$ so that $\zeta(z)$ may be single valued. Then $R$ is mapped onto the domain $D$ with enumerably infinite number of radial slits and $z=z(\zeta)$ can be continued analytically along radii from $\zeta=0$ to $|\zeta|=1$ except possibly a set of $\theta$ of angular measure zero.[9] Let $(p)_\zeta$ be the image of $p$ on $D$. Now $(p)_\zeta$ may consist of infinitely many points. Let $z^{-1}(\mathrm{C}(r,p))$ be the image of $C(r,p)$. Then $z^{-1}(C(r,p))$ is a domain (consisting of infinite) number of components in $D$). Let $p_1$ and $p_2$ be two points of $R+B$. Then $\operatorname{dist}(p_1,p_2)$ is the infinimum of all curves $l$ connecting $p_1$ with $p_2$. But the length of a segment on $R$ is the euclidean length of the image (consisting of infinitely many components) $z^{-1}(l)$ of $l$. Hence

7) Z. Kuramochi: Singular points of Riemann surfaces.

8) Z. Kuramochi: Dirichlet problem on Riemann surfaces. IV, Proc. Japan Acad., **30**, 946–950(1954).

9) Z. Kuramochi: Harmonic measures and capacity of sets of the ideal boundary. II, Proc. Japan Acad., **31**, 25–30 (1955).

$\left|\frac{dl}{ds}\right|\leqq 1: ds=(d\xi+d\eta)^{\frac{1}{2}}: \zeta=\xi+i\eta=re^{i\theta}$. Put $g(z)=1-\frac{\text{dist}(z,p)-r_1}{r_2-r_1}$ in $C(r_2,p)-C(r_1,p)$, $g(z)=0$ in $R-C(r_2,p)$ and $g(z)=1$ in $C(r_1,p)$. Then $g(z)$ is continuous and

$$D(g(z))\leqq\iint\left(\left|\frac{dg(z)}{ds}\right|^2\left|\frac{ds}{d\xi}\right|^2+\left|\frac{dg(z)}{ds}\right|^2\left|\frac{ds}{d\eta}\right|^2\right)d\xi\, d\eta\leqq\frac{\pi}{(r_2-r_1)^2}.$$

Put $CV_n=0$. Then $g(z)$ is the required function. Hence Green's topology is D. S.

3). *For N-Martin's topology.* Put $G=C(r_2,p)$ and put $\Omega_{1-\varepsilon}=E[z\in R:\omega(\partial G,z)>1-\varepsilon]$. Assume $\lim_{\varepsilon\to 0}(\Omega_{1-\varepsilon}\cap C(r_1,p)\cap B,z)=\omega^*(z)>0$. Then $\omega^*(z)=\int_{C(r_1,p)\cap B_1^N} N(z,p)\,d\mu(p)$ and by Theorem 13 of $P$[10] we have by $\text{dist}(CG,C(r_1,p))>0$ $\omega^*(z)>{}_{CG}\omega^*(z)$, where ${}_{CG}\omega^*(z)$ is the function such that ${}_{CG}\omega^*(z)=\omega^*(z)$ on $CG$ and ${}_{CG}\omega^*(z)$ has M.D.I. over $G$. Put $\omega'(z)=\omega^*(z)-{}_{CG}\omega^*(z)$ and $M=\sup\omega'(z)$. Since $\omega^*(z)$ and ${}_{CG}\omega^*(z)$ are $\overline{\text{harmonic}}$[11] in $C(r_2,p)-(C(r_1,p)\cap\Omega_{1-\varepsilon})-V_M: V=E\left[z\in R:\omega'(z)>\frac{M}{2}\right]$, by the $\overset{*}{\text{m}}$aximum[12] principle

$$\omega^*(z)-{}_{CG}\omega^*(z)\leqq\frac{M}{2}\omega(CV_M\cap\Omega_{1-\varepsilon},z)+\omega(V_M\cap\Omega_{1-\varepsilon}\cap C(r_1,p),z).$$

If $\omega(V_M\cap\Omega_{1-\varepsilon}\cap C(r_1,p),z)=0$ held, $\sup\omega^*(z)\leqq\frac{M}{2}$. This is a contradiction.

Hence $$\lim_{\varepsilon\to 0}\omega(V_M\cap\Omega_{1-\varepsilon}\cap C(r_1,p),z)>0.$$

Next by the Dirichlet principle

$D(\omega(V_M\cap\Omega_{1-\varepsilon}\cap C(r_1,p),z))\leqq D(\omega(V_M\cap\Omega_{1-\varepsilon}\cap C(r_1,p),z,C(r_2,p))$

$\leqq D\left(\frac{2\omega'(z)}{M}\right)<\infty$, because $\frac{2\omega'(z)}{M}\geqq 1$ in $V_M$ and $\frac{2\omega'(z)}{M}=0$ on $\partial C(r_2,p)$.

Hence $$\lim_{\varepsilon\to 0}\omega(V_M\cap\Omega_{1-\varepsilon}\cap C(r_1,p),z,C(r_2,p))=\omega^{**}(z)>0.$$

Clearly $\omega^{**}(z)\leqq\omega^*(z)$. Consider the regular niveau curves of $\omega^{**}(z)$. Then we have similarly as $C$) of Theorem 12 of $P$

$$D(\omega^{**}(z))-\delta_0\geqq\int_{C_\delta}\omega^*(z)\frac{\partial}{\partial n}\omega^{**}(z)\,ds=\int_{C_{1-\varepsilon}}\omega^*(z)\frac{\partial}{\partial n}\omega^{**}(z)\,ds$$
$$<\int_{C_{1-\varepsilon}}\omega^{**}(z)\frac{\partial}{\partial n}\omega^{**}(z)\,ds=(1-\varepsilon)D(\omega^{**}(z))$$

for a constant $\delta_0>0$ and for any constant $\varepsilon>0$.

This contradicts $\omega^{**}(z)\leqq\omega^*(z)$ for $\varepsilon<\frac{\delta_0}{2}$. Hence

10) We abbreviate "Potentials on Riemann surfaces" and "Singular points" by $P$ and $S$ respectively.

11) See $P$.

12) See $P$.

$$\lim_{\varepsilon\to 0}\omega(\Omega_{1-\varepsilon}\cap C(r_1,p)\cap B,z)=0. \tag{1}$$

By $\omega(CG,z)\geqq w(CG,z,R-R_0)$ $E[z\in R: w(CG,z,R-R_0)>1-\varepsilon]=\Omega^W_{1-\varepsilon}\subset\Omega_{1-\varepsilon}$ and we have $\lim_{\varepsilon\to 0} w(\Omega^W_{1-\varepsilon}\cap C(r_1,p)\cap B,z,R-R_0)=0$. Now $R$ is a surface with positive boundary, whence $\lim_{\varepsilon\to 0} w(\Omega^W_{1-\varepsilon}\cap C(r_1,p)\cap B,z)=0$. Thus $N$-Martin's topology is H.S.

Suppose $\omega(C(r_1,p)\cap B,z)>0$. Put $\Omega_n=E\left[z\in R:\omega(\partial G,z)>1-\frac{1}{n}\right]$. Then by $\omega(C(r_1,p)\cap B,z)\leqq 1=\omega(\partial G,z)$ on $\partial G$ we have ${}_{CG}\omega(C(r_1,p)\cap B,z)\leqq\omega(\partial G,z)$ in $G$, where $G=C(r_2,p)$. Hence

$$\Omega_n\subset C\widetilde{V}_n=E\left[z\in R: {}_{CG}\omega(B\cap C(r_1,p),z)>1-\frac{1}{n}\right]. \tag{2}$$

Put $\mho_n=E\left[z\in R:\omega(C(r_1,p)\cap B,z)>1-\frac{1}{2n}\right]$. Then by P.C.1 $\omega(C(r_1,p)\cap BC\mho_n,z)=0$ for every $n$. Put $V_n=\widetilde{V}_n\cap\mho_n$. Then by (1) and (2) we have

$$\omega(CV_n\cap B\cap C(r_1,p),z)\leqq\omega(C\mho_n\cap B\cap C(r_1,p),z)$$
$$+\omega(C\widetilde{V}_n\cap C(r_1,p)\cap \mathrm{B},z)=\omega(C\widetilde{V}_n\cap B\cap C(r_1,p),z)\downarrow 0 \text{ as } n\to\infty. \tag{3}$$

On the other hand, by $\omega(C(r_1,p)\cap B,z)>0$ and by (3) we have

$$\omega(V_n\cap B\cap C(r_1,p),z)>0. \tag{4}$$

Now $\omega'(z)=\omega(C(r_1,p)\cap B,z)-{}_{CG}\omega(C(r_1,p)\cap B,z)\geqq\frac{1}{2n}$ in $V_n$, $\omega'(z)=0$ on $\partial G$ and $D(\omega'(z))<\infty$. Hence $U(z)=\min(1,2n\omega'(z))$ is a function required. Thus by (3) and (4) $N$-Martin's topology is D.S.

**Lemma 7.** *Let $R$ be a Riemann surface with positive boundary and with K-Martin's topology. Let $A$ be a closed set in $B$ such that $w(A,z)>0$. Put $F=E[z\in R:\mathrm{dist}(z,A)>\delta_0>0]$. Then $w_{B\cap F}(A,z)=\lim_n w_{B_n\cap F}(A,z): B_n=E\left[z\in\overline{R}:\mathrm{dist}(z,B)\leqq\frac{1}{n}\right]$.*

In fact, by $w_A(A,z)=w(A,z)$ it can be proved similarly as Theorem 13 of $P$ that $w(A,z)=\int\limits_{A\cap(\overline{R}-B^K_0)} K(z,p)\,d\mu(p)$ and by $\mathrm{dist}(F,A)>0$ ${}_{F\cap B}(w_{F\cap B}(A,z))=w_{F\cap B}(A,z)<w(A,z)$, where $B^K_0$ is the set of $K$-non minimal point. Now $w_{F\cap B}(A,z)$ is harmonic and $w(z)=w(A,z)-w_{F\cap B}(A,z)>0$ is also superharmonic in $R$. Let $\mu'$ be the canonical mass distribution of $w(z)$. Now by $w_{F\cap B}(A,z)={}_{F\cap B}(w_{F\cap B}(A,z))$ $w_{F\cap B}(A,z)$ is represented by a canonical mass $\mu''$ on $B^K_1\cap F: B^K_1=B^K-B^K_0$. Hence $\mu=\mu'+\mu''$. But $\mu=0$ on $F$. Hence if $\mu''>0$, this contradicts the uniqueness of canonical mass distribution. Hence $\mu=0$, i.e. $w_{F\cap B}(A,z)=0$.

4). *For K-Martin's topology.* Put $\Omega_{1-\varepsilon}=E[z\in R: w(\partial G, z)>1-\varepsilon]$. Then $w(\Omega_{1-\varepsilon}\cap B\cap C(r_1, p), z)\leqq\dfrac{w_{C(r_1,p)\cap B}(G,z)}{1-\varepsilon}=0$ by $\mathrm{dist}(\partial G, C(r_1,p))>0$. Thus $K$-Martin's topology is H. M.