39. *On a Variant of Hausdorff Measure-Bend*

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1. Conditions for countable straightenableness and countable rectifiability. The present article is a continuation of our recent notes which have appeared in these Proceedings. The underlying space $\boldsymbol{R}^m$ will be assumed throughout to be at least two-dimensional.

We begin by stating the following result which is analogous to Theorem (9.1) on p. 233 of Saks [6] and which may be established as for that theorem with the aid of the category theorem of Baire.

THEOREM. *In order that a curve which is continuous on a nonvoid closed set E of real numbers, be countably straightenable [or countably rectifiable] on E* (see [5]§4 and [1]§2), *it is necessary and sufficient that every nonvoid closed subset of E should contain a portion on which the curve is straightenable [rectifiable].*

There is another condition sufficient for countable rectifiability which is closely related to Theorem (10.8) of Denjoy given on p. 237 of Saks [6]. For this purpose we have to introduce a few definitions. A curve φ situated in $\boldsymbol{R}^m$ will be said to be *conic on the right [on the left]* at a point $t_0 \in \boldsymbol{R}$, iff (i.e. if and only if) it is possible to choose a number δ of the interval $0<\delta<\pi/2$ and a nonvanishing vector p of the space $\boldsymbol{R}^m$, in such a manner that whenever a closed interval I of length $|I|<\delta$ has t_0 for its left-hand [right-hand] extremity and moreover the increment $\varphi(I)$ of the curve φ over I does not vanish, the angle between $\varphi(I)$ and p is less than $(\pi/2)-\delta$. We shall further term φ to be *unilaterally conic* at t_0 iff it is conic on the side (right or left) at t_0.

Our condition may now be set forth in the following form.

THEOREM. *If at every point t of a linear set E, except perhaps at the points of a countable subset, a curve φ is unilaterally conic, then φ is countably rectifiable on E.*

PROOF. Let A be the set of the points of $\boldsymbol{R}$ at which the curve φ is conic on the right. It is certainly enough to show that φ is countably rectifiable on A. Consider the rational vectors (i.e. having rational components) of $\boldsymbol{R}^m$ other than the zero vector. Noting that they are countable in number, we arrange all of them in a distinct infinite sequence $p_1, p_2, \cdots$. For each natural number n we denote by A_n the set of the points $t \in \boldsymbol{R}$ such that $|\varphi(t)|<n$ and further that, for every closed interval I whose length is $<1/n$ and whose left-hand extremity is t, the condition $\varphi(I)\neq 0$ implies the inequality $\varphi(I)\diamond p_n<(\pi/2)-(1/n)$. We then obtain easily $A=A_1\cup A_2\cup\cdots$, and so the proof reduces to ascertaining that φ is countably rectifiable on each A_n. For later purpose we remark in passing that for every point t of A_n we have $|\varphi(t)p_n|\leqq|\varphi(t)|\cdot|p_n|<n|p_n|$, where $\varphi(t)p_n$ means the scalar product of $\varphi(t)$ and p_n.

Keeping n fixed, let us write $A_n^{(k)}=A_n\cdot(k/n,\ (k+1)/n]$ for each integer k (positive or not), so that A_n is the union of the sets $A_n^{(k)}$ for all k. If now J is any closed interval whose extremities belong to $A_n^{(k)}$ (and *a fortiori* to A_n), we find at once, in view of the definition of the set A_n, that $|\varphi(I)|\cdot|p_n|\cdot\sin(1/n)\leqq\varphi(I)p_n$. Hence, however we may extract from $A_n^{(k)}$, where k is fixed, a finite sequence of points $t_1<\cdots<t_j$, we have

$$L(\varphi;\{t_1,\cdots,t_j\})\cdot|p_n|\cdot\sin(1/n)\leqq\{\varphi(t_j)-\varphi(t_1)\}p_n<2n|p_n|,$$

the last step being effected by the inequality $|\varphi(t)p_n|<n|p_n|$ already mentioned. This shows us that the length $L(\varphi;\{t_1,\cdots,t_j\})$ is bounded upwards. Since the sequence $t_1<\cdots<t_j$ is arbitrary, it follows immediately that the curve φ is rectifiable on the set $A_n^{(k)}$. This implies finally the countable rectifiability of φ over A_n, and the proof is complete.

REMARK. It might be possible to obtain a result similar in its character to our second theorem and stating a sufficient condition for a curve (not necessarily continuous) to be Borel-rectifiable ([2]§1]) over a linear set. On the other hand it is permitted to replace in our first theorem the word "countably" by "Borel" throughout. In point of fact, countable straightenableness [countable rectifiability] of a curve over a linear set on which it is continuous is equivalent to Borel straightenableness [Borel rectifiability] of the curve over the same set (see [5]§4).

2. A case in which the Hausdorff and reduced measure-bends of a curve coincide on a set.

THEOREM. *If a curve φ is B-straightenable on a set E, then*

$$\Pi(\varphi;E)=\Upsilon(\varphi;E).$$

PROOF. Since, in abridged notations, $\Pi(E)\leqq\Upsilon(E)$ by the theorem of [5]§2, we need only derive the converse inequality. The set E, which we may assume nonvoid, admits by hypothesis an expression as the union of a disjoint sequence Δ of bounded sets which are relatively Borel in E and on each of which φ is straightenable. We then have both $\Pi(E)=\Pi(\Delta)$ and $\Upsilon(E)=\Upsilon(\Delta)$, since the Hausdorff and reduced measure-bends of a curve are always outer measures in the sense of Carathéodory. Without loss of generality we may therefore suppose E bounded and φ straightenable on E.

We inspect now the proof for the lemma of [3]§1 and find that it is possible to decompose E into a finite disjoint sequence of sets, say $\varDelta_0=\langle E_1,\cdots,E_n\rangle$, such that every E_i is a relative Borel set in E $(i=1,\cdots,n)$ and fulfils the inequality $\Omega(E_i)<\pi/2$. (It should be noticed that the boundedness of φ on E is unnecessary for the construction of such a sequence $\varDelta_0$.) It follows that $\Pi(E)=\Pi(\varDelta_0)$ and similarly for Υ. We may thus assume in addition that $\Omega(E)<\pi/2$.

This being so, express E in any manner as the join of a sequence Θ of its subsets. Noting that $\Omega(N)<\pi/2$ when N is a set in Θ, we apply the theorem of [5]§3 and obtain $\Phi(N)=\omega(\varphi[N])=\Omega(N)$ for each N, so that $\Phi(\Theta)=\Omega(\Theta)\geqq\Upsilon(\Theta)$. Remember now the definition of Hausdorff measure-bend (see [5]§2), and we find at once $\Pi(E)\geqq\Upsilon(E)$, which completes the proof.

3. A quantity resembling Hausdorff measure-bend. Given a curve φ, we shall retain the notation $\Phi(E)=\omega(\varphi[E])$ at the end of the foregoing section, E being any linear set. Similarly we shall write $\Phi_0(E)=\omega_0(\varphi[E])$, where $\omega_0(X)$ denotes for any $X\subset\boldsymbol{R}^m$ the outer bend of X (see [2]§5).

In [5]§2 we have defined $\Pi(\varphi;E)$ by a limiting process, with the aid of the set-function ω. If we now use ω_0 in place of ω and perform the same limiting process, we obtain a geometric quantity analogous to $\Pi(\varphi;E)$. This will be denoted by $\Pi_0(\varphi;E)$. In other words, given a positive number ε, we express E as the union of an arbitrary sequence $\varDelta$ of sets with diameters less than ε and consider the infimum of $\Phi_0(\varDelta)$ for all choices of $\varDelta$; the limit, as $\varepsilon\to0$, of this infimum is then $\Pi_0(\varphi;E)$ by definition. It is easily verified that $\Pi_0(\varphi;E)$, *qua function of E, is an outer Carathéodory measure which vanishes when E is countable.*

As we shall see below, there are cases in which $\Pi_0(\varphi;E)$ turns out equal to $\Pi(\varphi;E)$. But we are not in a position to decide whether or not the two quantities are completely identical in all cases.

LEMMA. *We have $\Pi_0(\varphi;E)\geqq\Pi(\varphi;E)$ for any curve φ and any set E.*

PROOF. Suppose $\Pi_0(\varphi;E)$ finite and consider any positive number ε. It is plainly possible to express E as the join of an infinite sequence of sets $E_1,E_2,\cdots$ with diameters less than ε, such that $\sum\Phi_0(E_n)<A_0+\varepsilon$, where and subsequently A_0 is short for $\Pi_0(\varphi;E)$ and n ranges over $1,2,\cdots$. For each n we can express the set E_n as the join of a sequence $\varDelta_n$ of sets, in such a manner that $\sum\Phi(\varDelta_n)<A_0+\varepsilon$. But it is evident that $\Pi_\varepsilon(\varphi;E)\leqq\sum\Phi(\varDelta_n)$, with the same meaning for the left-hand side as in [5]§2. We thus get $\Pi_\varepsilon(\varphi;E)<A_0+\varepsilon$. Hitherto ε has been kept fixed. We make now $\varepsilon\to0$ and obtain at once $\Pi(\varphi;E)\leqq A_0$, completing the proof.

THEOREM. *If a curve φ is continuous on a set E, we have*

$$\Pi_0(\varphi; E)=\Pi(\varphi; E)\geqq\Phi_0(E).$$

PROOF. 1) The inequality. Let us write for short $A=\Pi(\varphi; E)$. To prove $A\geqq\Phi_0(E)$, we may suppose A finite and E nonvoid. Continuity of φ on E implies that, given any $\varepsilon>0$, each point t of E can be enclosed in an open interval $I(t)$ with rational extremities and such that $\mathrm{d}(\varphi[EI(t)])<\varepsilon$. We can clearly extract from E an infinite sequence of (not necessarily distinct) points $t_1, t_2, \cdots$ so that the intervals $I_n=I(t_n)$, where $n=1, 2, \cdots$, together cover E. Then E is decomposed into a disjoint infinite sequence of sets $E_1, E_2, \cdots$ which are defined by $E_1=EI_1$ and

$$E_{n+1}=E(I_{n+1}-I_1-\cdots-I_n) \quad (n=1, 2, \cdots);$$

so that $\mathrm{d}(\varphi[E_n])<\varepsilon$ for every n and moreover $A=\sum\Pi(\varphi; E_n)$. For each n, on the other hand, E_n may be expressed as the join of a sequence Δ_n of sets such that $\Phi(\Delta_n)<\Pi(\varphi; E_n)+2^{-n}\varepsilon$. By summing this over all n we derive $\sum\Phi(\Delta_n)<A+\varepsilon$. The last inequality shows that E admits an expression as the join of an infinite sequence of sets $M_1, M_2, \cdots$, such that $\mathrm{d}(\varphi[M_n])<\varepsilon$ for each n and that $\sum\Phi(M_n)<A+\varepsilon$. Noting that the images $\varphi[M_n]$ together make up $\varphi[E]$, we let $\varepsilon\to 0$ and readily deduce $\Phi_0(E)=\omega_0(\varphi[E])\leqq A$, as required.

2) The equality of the assertion will be reduced to the inequality just established. In the first place we see by our lemma that it suffices to derive $\Pi_0(\varphi; E)\leqq A=\Pi(\varphi; E)$. Given any $\varepsilon>0$, we decompose the whole line $\boldsymbol{R}$ into a disjoint infinite sequence Δ of half-open intervals $J_1, J_2, \cdots$ with lengths less than ε, so that $A=\Pi(\varphi; E\Delta)$. But we must have $\Phi_0(EJ_n)\leqq\Pi(\varphi; EJ_n)$ for every n; for we may plainly replace the set E by EJ_n in our inequality $\Phi_0(E)\leqq\Pi(\varphi; E)$. It follows at once that $\Phi_0(E\Delta)\leqq A$. Since ε is arbitrary, this implies directly that $\Pi_0(\varphi; E)\leqq A$, which completes the proof.

THEOREM. (i) *If a curve φ is B-rectifiable on a set E, we have* $\Pi_0(\varphi; E)=\Pi(\varphi; E)$; (ii) *if on the other hand φ is B-straightenable on E, it is B-rectifiable on E.*

PROOF. *re* (i): By hypothesis, the set E can be covered by a disjoint sequence Δ of Borel sets on whose intersections with E the curve φ is rectifiable. Then $\Pi_0(\varphi; E)=\Pi_0(\varphi; E\Delta)$ and $\Pi(\varphi; E)=\Pi(\varphi; E\Delta)$. Without loss of generality we may therefore assume further φ rectifiable on E. This being so, consider a rectifiable curve ψ which coincides on E with φ. Then $\Pi_0(\varphi; E)=\Pi_0(\psi; E)$ and similarly for Π, so that it is enough to derive $\Pi_0(\psi; E)=\Pi(\psi, E)$. Let now H be the set of all the points of discontinuity for ψ. Since ψ is rectifiable, H must be countable. Accordingly $\Pi_0(\psi; E)=\Pi_0(\psi; E-H)$, and similarly for Π. But the curve ψ is continuous on $E-H$, and so $\Pi_0(\psi; E-H)$ equals $\Pi(\psi; E-H)$ in virtue of the foregoing theorem. Hence the result.

re (ii): It is sufficient to show that a curve φ is Borel-rectifiable on a set $X\subset \boldsymbol{R}$ whenever it is straightenable on X. For this purpose we define a linear set T as follows: a point t belongs to T iff t is a point of accumulation for X and further, given any open interval I containing t, the curve φ is unbounded on the intersection IX. Then T must be a finite set, as we found in the course of the proof for the theorem of [4]§2. On the other hand each point of $\boldsymbol{R}-T$ can be enclosed, by definition of T, in some open interval J with rational endpoints and such that φ is bounded on the intersection JX. But φ is then rectifiable on JX on account of the lemma of [3]§1. Since there is only a countable infinity of open intervals with rational extremities, we conclude that φ is Borel-rectifiable on the set X.

REMARK. The part for statement (ii) of the above proof may also be attached to the following proposition: *a curve is countably rectifiable on a set whenever it is countably straightenable on the same set.*

4. Multiplicity function. Given a curve φ and a set $E\subset \boldsymbol{R}$, we define as before the multiplicity function $N(\varphi; x; E)$, where $x\in \boldsymbol{R}^m$, to be the number (finite or $+\infty$) of the points t of E such that $\varphi(t)=x$.

THEOREM. *If E is a Borel set and φ is B-straightenable on E in the above, the function $N(\varphi; x; E)$ is measurable with respect to the outer bend ω_0 and we have the relation*

$$\Upsilon(\varphi; E)=\Pi(\varphi; E)=\Pi_0(\varphi; E)=\int_{\mathbf{R}} N(\varphi; x; E)d\omega_0(x).$$

PROOF. We may restrict ourselves to the last equality, for the first two equalities are already obtained in the foregoing two sections. To shorten our notations, we shall write $\Pi_0(M)$ and $N(x; M)$ for $\Pi_0(\varphi; M)$ and $N(\varphi; x; M)$ respectively, M being any linear set. It is obvious that if we decompose the set E into a (disjoint) sequence Δ of Borel sets, then $\Pi_0(E)=\Pi_0(\Delta)$ and $N(x; E)=N(x; \Delta)$ for every $x\in \boldsymbol{R}^m$. This, combined with part (ii) of our last theorem, allows us to assume φ rectifiable on E. There then exists a rectifiable curve coinciding on E with φ, and it follows at once that we may suppose φ itself rectifiable (over $\boldsymbol{R}$). If, consequently, A denotes the set of all the points of E at which φ is discontinuous, A is countable and hence so must be its image $\varphi[A]$ also. Then $N(x; A)$, which is zero unless $x\in\varphi[A]$, is measurable (ω_0) and its integral (ω_0) vanishes, where and below integration is always extended over the whole space $\boldsymbol{R}^m$. Further, we clearly have $\Pi_0(A)=0$. On writing $B=E-A$, our task therefore comes to proving the measurability (ω_0) of $N(x; B)$ and the equality $\alpha=\Pi_0(B)$, where α abbreviates the integral (ω_0) of $N(x; B)$. We observe in passing that φ is continuous at all points of the set B.

Given a natural number n, let us consider the half-open intervals $I_n^{(k)}=(k/2^n,(k+1)/2^n)$ for $k=0,\pm 1,\pm 2,\cdots$ and arrange them in a sequence Δ_n. Then Δ_{n+1} is a refinement of Δ_n for each n, and if $F_n(x)$ means the sum, for all values of k, of the characteristc functions of the images $\varphi[BI_n^{(k)}]$, it is seen that the functions $F_1(x), F_2(x),\cdots$ constitute a monotone non-decreasing sequence tending to $N(x; B)$. Furthermore each $\varphi[BI_n^{(k)}]$ is an analytic set in $\boldsymbol{R}^m$, since it is a continuous image of a Borel set. Now, as is well known, analytic sets are measurable with respect to any outer Carathéodory measure. It follows that each $F_n(x)$ is measurable (ω_0) and that its integral (ω_0) tends non-decreasingly to α (see above) as $n\to+\infty$. In other words, we have $\Phi_0(B\Delta_n)\uparrow\alpha\ (n\to+\infty)$, where we write as before $\Phi_0(M)=\omega_0(\varphi[M])$ when $M\subset\boldsymbol{R}$. But it is evident by definition of $\Pi_0(B)$ and by construction of the sequence Δ_n that $\Pi_0(B)$ cannot exceed the supremum of $\Phi_0(B\Delta_n)$ for all n. Accordingly we get $\Pi_0(B)\leqq\alpha$, and thus it only remains to verify the opposite inequality. Since φ is continuous on B, the first theorem of §3 requires that $\Phi_0(M)\leqq\Pi_0(M)$ whenever $M\subset B$. We therefore obtain $\Phi_0(B\Delta_n)\leqq\Pi_0(B\Delta_n)$ $=\Pi_0(B)$ for every n. Making $n\to+\infty$ here, we deduce at once $\alpha=\lim\Phi_0(B\Delta_n)\leqq\Pi_0(B)$, which completes the proof.

References

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