74. A Theorem on Weak Homotopy Equivalences

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1. **Introduction.** Let X and Y be spaces with base points x_0 and y_0 respectively. We denote by X^Y the mapping space of maps $(Y, y_0)\to(X, x_0)$ with the compact-open topology; the base point is a constant map $Y\to x_0$. Any map $g: (X, x_0)\to(Z, z_0)$ of a space $X\ni x_0$ into another space $Z\ni z_0$ induces a map $'g: X^Y\to Z^Y$ of X^Y into Z^Y defined by $['g(f)](y)=(gf)(y)$ for $f\in X^Y$ and $y\in Y$. Then we have

Theorem 1. *Let spaces $E\supset F$, $B\supset C$ and a map $p: (E,F)\to(B,C)$ be given. If p is a weak homotopy equivalence of pairs, i.e. if p induces an isomorphism*

$$p_*: \pi_n(E,F)\approx\pi_n(B,C) \quad \text{for any } n\geqq 0,$$

then for any CW-complex K the induced map $'p: (E^K, F^K)\to(B^K, C^K)$ is a weak homotopy equivalence of pairs, i.e. $'p$ induces an isomorphism

$$'p_*: \pi_n(E^K, F^K)\approx\pi_n(B^K, C^K) \quad \text{for any } n\geqq 0,$$

where we mean a 1–1 correspondence by an isomorphism if $n\leqq 1$.

This will be proved in section 3 by using Sugawara's homotopically covering homotopy theorem ([5], Theorem 3) and Morita's theorem concerning an exponential law for mapping spaces ([4], Theorem 6).

In the next paper we intend to apply this theorem to establish a generalization of Dold-Thom's isomorphism theorem ([1], Satz 6. 10) for the homotopy groups with coefficients in the sense of Katuta [2] (cf. [3]).

2. **Preliminaries.** Throughout this paper we mean by a *space* a topological space with base point, by a *map* a continuous map which carries the base point to the base point and by a *homotopy* a homotopy relative to the base point.

Let X and Y be Hausdorff spaces and let Z be any space. With any map $f: X\times Y\to Z$ there is associated a map $f': Y\to Z^X$ by the formula $[f'(y)](x)=f(x, y)$ for $y\in Y$ and $x\in X$. The correspondence $f\to f'$ defines a 1–1 map

$$\theta: Z^{X\times Y}\to(Z^X)^Y$$

K. Morita [4] has introduced the following notion. It is said that a Hausdorff space W *has a weak topology with respect to compact sets in the wider sense* if a subset A of W such that $A\frown K$ is closed for every compact set K of W is necessarily closed. For instance, if X is a CW-complex and Y is a locally compact Hausdorff space, then $X\times Y$ has the above property. K. Morita proved the following result in [4].

(2.1) *If $X\times Y$ is a Hausdorff space having the weak topology with respect to compact sets in the wider sense, then the map*

$$\theta: Z^{X\times Y}\to(Z^X)^Y$$

is a homeomorphism onto. If, in addition, $A\subset X, B\subset Y, C\subset Z$, then θ induces a homeomorphism onto:

$$\theta_1: (Z, C)^{(X\times Y, A\times Y\cup X\times B)}\to((Z, C)^{(X,A)}, C^X)^{(Y,B)}.$$

Let X and Y be spaces with base points x_0 and y_0 respectively. We denote by $X\# Y$ the space obtained from $X\times Y$ by contracting the subspace $X\vee Y=X\times y_0\cup x_0\times Y$ to a point; the base point in $X\# Y$ is the image of $X\vee Y$. Then we easily obtain from (2.1)

(2.2) *Under the assumption of* (2.1) *the induced map*

$$\theta_2: Z^{X\# Y}\to(Z^X)^Y$$

is a homeomorphism onto. If, in addition, $A\subset X, B\subset Y, C\subset Z$, then θ_2 induces a homeomorphism onto:

$$\theta_3: (Z, C)^{(X\# Y, A\# Y\cup X\# B)}\to((Z, C)^{(X,A)}, C^X)^{(Y,B)}.$$

Clearly, we have the naturality of the map θ; that is,

(2.3) *The commutativities hold in the following diagrams:*

$$\begin{array}{ccc} Z^{X\times Y} & \xrightarrow{\theta} & (Z^X)^Y \\ \varphi^{\#}\uparrow & & \uparrow\varphi^{\#} \\ Z^{X'\times Y} & \xrightarrow{\theta} & (Z^{X'})^Y \end{array} \qquad \begin{array}{ccc} Z^{X\times Y} & \xrightarrow{\theta} & (Z^X)^Y \\ \psi^{\#}\uparrow & & \uparrow\psi^{\#} \\ Z^{X\times Y'} & \xrightarrow{\theta} & (Z^X)^{Y'} \end{array} \qquad \begin{array}{ccc} Z^{X\times Y} & \xrightarrow{\theta} & (Z^X)^Y \\ \eta_{\#}\downarrow & & \downarrow\eta_{\#} \\ Z'^{X\times Y} & \xrightarrow{\theta} & (Z'^X)^Y \end{array}$$

where $\varphi^{\#}, \psi^{\#}$ and $\eta_{\#}$ are maps induced by $\varphi: X\to X'$, $\psi: Y\to Y'$ and $\eta: Z\to Z'$ respectively in obvious ways.

Analogous properties to (2.3) are valid for induced maps θ_1, θ_2 and θ_3. Hereafter, for simplicity, we write θ instead of $\theta_i (i=1,2,3)$.

Let $E\supset F$ and $B\supset C$ be spaces and $p: (E, F)\to(B, C)$ be a map of pairs. Consider the following conditions (I) and (II), concerning such a map p (cf. [5]).

(I) *Let K be any CW-complex, L a subcomplex of K, $I=[0,1]$ (the closed unit interval), and M a subcomplex of the product complex $K\times I$. Let $N=((K\times 0)\cup(L\times I))\cap M$. Let two maps ξ and η be given such that in the diagram*

$$\begin{array}{ccc} ((K\times 0)\cup(L\times I), N) & \xrightarrow{\xi} & (E, F) \\ i\downarrow & & \downarrow p \\ (K\times I, M) & \xrightarrow{\eta} & (B, C) \end{array}$$

the two composite maps $p\xi$ and ηi are homotopic to each other by a homotopy of pairs

$$G: (((K\times 0)\cup(L\times I))\times I, N\times I)\to(B, C)$$

with $G(z,0)=p\xi(z)$, $G(z,1)=\eta i(z)$ *for* $z\in(K\times 0)\cup(L\times I)$, *where* i *is the inclusion map.*

From these assumptions, it follows that there exist a map

$$\lambda\colon (K\times I, M)\to(E,F)$$

and a homotopy

$$H\colon (K\times I\times I, M\times I)\to(B,C)$$

such that $\lambda i=\xi$, $H(z,0)=p\lambda(z)$, $H(z,1)=\eta(z)$ *for* $z\in K\times I$ *and* $H(z,t)=G(z,t)$ *for* $z\in(K\times 0)\cup(L\times I)$, $t\in I$.

(II) *In addition to the assumptions of* (I), *assume further that* $K=I^n(=I\times\cdots\times I$ *(n-times))*,[1] *with a cell structure such that it has only one n-cell* $I^n-\dot{I}^n$, $L=\dot{I}^n$ *and* $p\xi=\eta i$. *Then we have the conclusions of* (I), *i.e. there exist a map*

$$\lambda\colon (I^n\times I, M)\to(E,F)$$

and a homotopy

$$H\colon (I^n\times I\times I, M\times I)\to(B,C)$$

such that $\lambda i=\xi$, $H(z,0)=p\lambda(z)$, $H(z,1)=\eta(z)$ *for* $z\in I^n\times I$ *and* $H(z,t)=p\lambda(z)=\eta(z)$ *for* $z\in(I^n\times 0)\cup(\dot{I}^n\times I)$, $t\in I$.

M. Sugawara proved the following theorem ([5], Theorem 3).[2]

(2.4) *Let* $E\supset F$ *and* $B\supset C$ *be spaces and* $p\colon(E,F)\to(B,C)$ *be a map of pairs. Then the following statements are equivalent*:

(1) *p is a weak homotopy equivalence of pairs.*

(2) *p satisfies the condition* (I).

(3) *p satisfies the condition* (II).

3. **Proof of Theorem 1.** Assume that the induced map $'p$ satisfies the assumption of the condition (II); that is, let two maps ξ' and η' of pairs be given such that the following diagram

$$\begin{array}{ccc} (J^n, J^n\cap M) & \xrightarrow{\xi'} & (E^K, F^K) \\ \downarrow i & & \downarrow 'p \\ (I^{n+1}, M) & \xrightarrow{\eta'} & (B^K, C^K) \end{array}$$

is commutative (i.e. $'p\xi'=\eta' i$), where M is a subcomplex of $I^{n+1}=I^n\times I$, $J^n=(I^n\times 0)\cup(\dot{I}^n\times I)$ and i is the inclusion map. With the map η' we can associate a map

$$\eta\colon (K\times I^{n+1}, K\times M)\to(B,C)$$

defined by $\eta(k,x)=[\eta'(x)](k)$ for $k\in K, x\in I^{n+1}$. Since K is a CW-complex and I^{n+1} is a locally compact space, the correspondence $\eta'\to\eta$ is a homeomorphism between appropriate mapping spaces by (2.1). Similarly, with the map ξ' we associate a map

$$\xi\colon (K\times J^n, K\times(J^n\cap M))\to(E,F).$$

1) When $n=0$ we take I^0 for two points $\{0,1\}$ and $\dot{I}^0$ for the empty set.

2) M. Sugawara did not consider the case $n=0$. But (2.4) is easily verified in that case.

Then the following diagram is obviously commutative:

$$\begin{array}{ccc} (K\times J^n, K\times(J^n\frown M)) & \xrightarrow{\xi} & (E, F) \\ \downarrow{\scriptstyle 1\times i} & & \downarrow{\scriptstyle p} \\ (K\times I^{n+1}, K\times M) & \xrightarrow{\eta} & (B, C) \end{array}$$

Now $K\times I^{n+1}=(K\times I^n)\times I$, $K\times J^n=((K\times I^n)\times 0)\smile((K\times \dot{I}^n)\times I)$, $K\times(J^n\frown M)=(K\times J^n)\frown(K\times M)$ and $K\times I^n$ is a CW-complex, and hence the assumptions of (I) are satisfied. Since p is a weak homotopy equivalence, the conclusions of (I) hold by (2.4). Therefore there exist a map

$$\lambda: (K\times I^{n+1}, K\times M)\rightarrow(E, F)$$

and a homotopy

$$H: (K\times I^{n+1}\times I, K\times M\times I)\rightarrow(B, C)$$

such that $\lambda(1\times i)=\xi$, $H(z, 0)=p\lambda(z)$, $H(z, 1)=\eta(z)$ for $z\in K\times I^{n+1}$, and $H(z, t)=p\lambda(z)$ for $z\in K\times J^n, t\in I$. For λ and H we define a map

$$\lambda': (I^{n+1}, M)\rightarrow(E^K, F^K)$$

and a homotopy

$$H': (I^{n+1}\times I, M\times I)\rightarrow(B^K, C^K)$$

by $[\lambda'(x)](k)=\lambda(k, x)$ for $x\in I^{n+1}, k\in K$ and $H'(x, t)(k)=H(k, x, t)$ for $x\in I^{n+1}, t\in I, k\in K$ respectively. Then by (2.1) both maps $\lambda\rightarrow\lambda'$ and $H\rightarrow H'$ are homeomorphisms between appropriate mapping spaces. It is clear that λ' and H' satisfy the desired properties. Thus the map $'p$ satisfies the condition (II) and by (2.4) $'p$ is a weak homotopy equivalence.

Added in Proof. After the submission of the manuscript I have found a theorem of Spanier (Ann. of Math., **69**, 197 (1959)) which is closely related to our Theorem 1.

References

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