73. *On the Product of Some Quasi-Hausdorff and Logarithmic Methods of Summability*

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1. O. Szász [11] discussed the following problem concerning the product of two summability methods for sequences: If a sequence $\{s_n\}$ is summable by a regular T_1 method then is the T_2 transform of $\{s_n\}$, where T_2 is a regular sequence-to-sequence method, also summable by the T_1 method to the same sum as before? In what follows we denote $T_1 \cdot T_2$ as the iteration product of these two methods, that is the T_1 transform of the T_2 transform of a sequence. He answered this problem in the affirmative in the cases when

(a) Abel and Hausdorff summability,
(b) Laplace and Riesz summability,
(c) Borel and Hausdorff summability,
(d) Abel summability and the circle method,
(e) Abel summability and the S_α method.

He also gave an example of two regular methods, where T_1 does not imply $T_1 \cdot T_2$. (See [11, 12].) Here we denote "method A implies method B", when any sequence summable A is summable B to the same sum.

M. S. Ramanujan [9, 10] also answered this problem in the affirmative in the cases when

(f) Abel and quasi-Hausdorff summability for a bounded sequence.

(g) Borel and quasi-Hausdorff summability for a bounded sequence.

(h) Abel summability and the (S^*, μ) method for the sequence which satisfies some condition.

M. R. Parameswaran [4] answered this problem in the affirmative in the case when

(i) Nörlund summability and a method of the Nörlund type.

C. T. Rajagopal [5] and T. Pati [3] also proved several theorems concerning this problem.

D. Borwein [1] answered this problem in the affirmative in the case when

(j) logarithmic and Hausdorff summability.

The purpose of this note is to prove a theorem in the case when

(k) logarithmic and quasi-Hausdorff summability with some condition for a bounded sequence.

When a sequence $\{s_n\}$ is given we define the logarithmic method of summability as follows: If

$$\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$

tends to a finite limit s as $x\to 1$ in the open interval $(0,1)$, we say that $\{s_n\}$ is L-summable to s and write $\lim s_n = s(L)$. It is well known that the Abel method implies the L method. (See [2].) D. Borwein [1] studied this method in connection with the generalized Abel method.

On the other hand if

(1) $$h_n^* = \sum_{\nu\geqq n}\binom{\nu}{n} s_\nu \int_0^1 t^{n+1}(1-t)^{\nu-n}d\psi(t) \qquad (n=0,1,2,\cdots)$$

tend to a finite limit s as $n\to\infty$, we say that $\{s_n\}$ is quasi-Hausdorff summable to s and write $\lim s_n = s(H^*,\psi)$, where $\psi(t)$ is a function of bounded variation in the closed interval $[0,1]$. M. S. Ramanujan [6, 7, 8] investigated this method in complete detail. He proved that the (H^*,ψ) method is regular if and only if

(2) $$\psi(1)-\psi(0)=1. \text{ (See also [2].)}$$

In the paper [9] he proved the following

Theorem 1. *If $\{s_n\}$ is a bounded sequence and (H^*,ψ) is a regular quasi-Hausdorff method, then the Abel method A implies the $A\cdot(H^*,\psi)$ method.*

Here we prove the following

Theorem 2. *If we assume the same conditions as Theorem 1 and moreover*

(3) $$\int_0^\sigma \log t\,|d\psi(t)| \text{ is finite}$$

for a positive σ, then the L method implies the $L\cdot(H^*,\psi)$ method.

2. Proof of Theorem 2. For the proof we use the method of M. S. Ramanujan [9]. Since the quasi-Hausdorff transforms of $\{s_n\}$ are given by (1), we have

(4) $$\sum_{n=0}^{\infty}\frac{h_n^*}{n+1}x^{n+1} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\sum_{\nu\geqq n}\int_0^1 s_\nu\binom{\nu}{n}(1-t)^{\nu-n}t^{n+1}d\psi(t)$$

provided the right-hand member exists. To prove this existence we consider the right-hand member with s_ν replaced by $|s_\nu|$ and $\psi(t)$, supposed to be monotonic increasing (as is permissible). The right-hand member with these changes, is

$$\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\sum_{\nu\geqq n}\int_0^1 |s_\nu|\binom{\nu}{n}(1-t)^{\nu-n}t^{n+1}d\psi(t)$$
$$=\sum_{n}\frac{x^{n+1}}{n+1}\int_0^1\sum_{\nu\geqq n}|s_\nu|\binom{\nu}{n}(1-t)^{\nu-n}t^{n+1}d\psi(t)$$

$$=\int_0^1\sum_n\frac{x^{n+1}}{n+1}\sum_{\nu\geq n}|s_\nu|\binom{\nu}{n}(1-t)^{\nu-n}t^{n+1}d\psi(t)$$
$$=\int_0^1\sum_{\nu=0}^{\infty}|s_\nu|\sum_{n=0}^{\nu}\binom{\nu}{n}(1-t)^{\nu-n}\frac{x^{n+1}}{n+1}t^{n+1}d\psi(t)$$
$$=\int_0^1\sum_{\nu=0}^{\infty}|s_\nu|\,t\int_0^x(1-t+ut)^\nu du\,d\psi(t)$$
$$=\int_0^1\int_0^x\sum_\nu|s_\nu|(1-t+ut)^\nu du\cdot td\psi(t)$$

every inversion of operations being justified by the fact that we have only positive integrands or terms. The last integral is finite from the boundedness of $\{s_n\}$ and the condition (2).

Hence we get from (4)

$$(5)\qquad \sum_{n=0}^{\infty}\frac{h_n^*}{n+1}x^{n+1}=\int_0^1\int_0^x\sum_{\nu=0}^{\infty}s_\nu(1-t+ut)^\nu du\cdot td\psi(t).$$

Here we put

$$f(x)=\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}\text{ and }F(x)=\sum_{n=0}^{\infty}\frac{h_n^*}{n+1}x^{n+1}.$$

Since

$$\int_0^x\sum_{\nu=0}^{\infty}s_\nu(1-t+ut)^\nu du=\int_0^x f'(1-t+ut)du=\frac{1}{t}\{f(1-t+xt)-f(1-t)\},$$

we get from (5)

$$(6)\qquad F(x)=\int_0^1\{f(1-t+xt)-f(1-t)\}\,d\psi(t).$$

Substituting $x=1-\dfrac{1}{y}$, we have

$$g(y)=f\left(1-\frac{1}{y}\right)=f(x)\text{ and }G(y)=F\left(1-\frac{1}{y}\right)=F(x)$$

and from the assumption

$$\frac{-1}{\log(1-x)}f(x)\to s\quad\text{as}\quad x\to1-0$$

or

$$\frac{g(y)}{\log y}\to s\quad\text{as}\quad y\to\infty.$$

From (6) we have

$$(7)\qquad \frac{G(y)}{\log y}=\int_0^1\frac{g\left(\frac{y}{t}\right)}{\log y}d\psi(t)-\frac{1}{\log y}\int_0^1 f(1-t)d\psi(t)$$
$$=I-J,\text{ say.}$$

Since $\dfrac{g(y)}{\log y}\to s$ as $y\to\infty$, the same is true of $\dfrac{g\left(\frac{y}{t}\right)}{\log y/t}$ also, since, for $0<t\leq 1$, $\dfrac{y}{t}\geq y$. Hence

$$g\left(\frac{y}{t}\right)=\{s+o(1)\}\log\frac{y}{t} \quad \text{as} \quad \frac{y}{t}\to\infty .$$

We put

$$I=\int_0^1 \frac{s\log\dfrac{y}{t}}{\log y}\,d\psi(t)+o\left(\int_0^1 \frac{\log\dfrac{y}{t}}{\log y}\,d\psi(t)\right)=I_1+I_2 .$$

From (2) and (3)

$$I_1=s\int_0^1 \frac{\log y-\log t}{\log y}\,d\psi(t)=s-\int_0^1 \frac{\log t}{\log y}\,d\psi(t)$$

$$=s+o(1) \quad \text{as} \quad y\to\infty .$$

Similarly we get $I_2=o(1)$ as $y\to\infty$. Next we have

$$|J|\leq\frac{1}{\log y}\int_0^1 |f(1-t)||d\psi(t)|\leq\frac{M}{\log y}\int_0^1 \sum_{n=0}^{\infty}\frac{(1-t)^{n+1}}{n+1}|d\psi(t)| ,$$

where $|s_n|\leq M$ $(n=0,1,2,\cdots)$. Since

$$\sum_{n=0}^{\infty}\frac{(1-t)^{n+1}}{n+1}=-\log t \text{ for } 0<t\leq 1 ,$$

we get from (3)

$$|J|\leq\frac{M}{\log y}\int_0^1 (-\log t)|d\psi(t)|=o(1) \text{ as } y\to\infty .$$

Collecting above estimations we have

$$\frac{G(y)}{\log y}\to s \quad \text{as} \quad y\to\infty ,$$

which proves the theorem.

3. Remark. In the special case when

$$\psi(t)=0 \quad \text{for} \quad 0\leq t<r$$
$$=1 \quad \text{for} \quad r\leq t\leq 1,$$

which satisfies the conditions of our theorem, we have the circle method of summability (γ, r) for $0<r<1$. Then (4) and (5) become respectively

$$(4') \qquad \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\sum_{\nu\geq n}s_\nu\binom{\nu}{n}(1-r)^{\nu-n}r^{n+1}$$

and

$$(5') \qquad r\int_0^x \sum_{\nu=0}^{\infty}s_\nu(1-r+ur)^\nu du .$$

From the L summability of $\{s_n\}$ $\sum_{n=0}^{\infty}\dfrac{s_n}{n+1}x^{n+1}$ and $\sum_{n=0}^{\infty}s_nx^n$ converge absolutely in the interval $0\leq x<1$. So we can interchange the order of two summations in (4′) and get the equality (4′)=(5′) irrespective of whether $\{s_n\}$ is bounded or not. Thus (7) reduces to the following expression

$$\frac{G(y)}{\log y}=\frac{1}{\log y}\left\{g\left(\frac{y}{r}\right)-f(1-r)\right\}\to s \quad \text{as} \quad y\to\infty.$$

Hence we have the following

Corollary. *The L method implies the* $L\cdot(\gamma, r)$ *method for* $0<r<1$.

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