110. *On Cohomological Dimension for Paracompact Spaces. I*

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1. **Introduction.** In 1954 H. Cohen [1] defined a cohomological dimension for locally compact Hausdorff spaces and proved several properties which are desirable for a dimension function. His definition reads as follows: a locally compact Hausdorff space X has $cdX\leqq n$ if and only if for each closed set C of X and for each compact set K in X $H^m(K\frown C; G)$ is the image of $H^m(K; G)$ of the homomorphism induced by the inclusion of $K\frown C$ into K where m is any integer such that $m\geqq n$ and $H^m(K; G)$ and $H^m(K\frown C; G)$ are the m-th Čech cohomology groups with the non-zero additive Abelian group G as coefficient.

In this paper we will modify his definition for paracompact Hausdorff spaces and establish some properties: the monotone property and the sum theorem.

All spaces in the present paper will be assumed to be paracompact Hausdorff unless otherwise specified, and all coefficient groups will be assumed to be non-zero additive Abelian groups.

2. **Preliminaries.** *Notations.* Let X be a space and let A be a closed set of X. For each non-negative integer n $H^n(X; G)$ means the n-th Čech cohomology group of X with G as coefficient and $H^n(X, A; G)$ means the n-th Čech cohomology group of X relative to A. If e is an element of $H^n(X; G)$, then we denote by $e|A$ the image of e by the homomorphism of $H^n(X; G)$ into $H^n(A; G)$ induced by inclusion.

Let α, β be open coverings of X. Since X is paracompact, all open coverings are assumed to be locally finite. For each open covering α of X we let $N(\alpha)$ be the nerve of α. If β is a refinement of α, then there is a projection Π of $N(\beta)$ into $N(\alpha)$ and this Π induces the homomorphisms $\Pi_{\alpha\beta}$ of n-cochain group, n-cocycle group and n-cohomology group of $N(\alpha)$ into n-cochain group, n-cocycle group and n-cohomology group of $N(\beta)$, respectively.

If $t^n=(U_0,\cdots, U_n)$ is an n-simplex of $N(\alpha)$, then we denote by $(t^n)_0$ the set $\bigcap_{k=0}^{n} U_k$. If $c^n=\sum_{\mu} a_\mu t^n_\mu$ is an n-cochain of $N(\alpha)$ (where each a_μ is a non-zero element of G), then we denote by $(c^n)_0$ the set

$\bigcup_{\mu}(t^n_\mu)_0$. Moreover, for each n-cochain $c^n=\sum_{\mu} a_\mu t^n_\mu$ of $N(\alpha)$ and for each closed set A of X let $c^n|A$ be the subcochain of c^n such as $\sum_{\nu}\{a_{\mu_\nu} t^n_{\mu_\nu} \mid (t^n_{\mu_\nu})_0 \frown A \neq \phi\}$. ($\phi$ is the empty set.)

Let α and β be two collections of open sets of X. We denote by $\alpha \smile \beta$ the collection $\{U \mid U \in \alpha$ or $U \in \beta\}$. If A is a set of X and if α is a collection of sets of X, then we denote by $\alpha|A$ the collection $\{U \frown A \mid U \in \alpha\}$.

Let A and B be two sets of X such as $A \subset B$. Let α and β be two locally finite collections of (open) sets of X such that α covers A and β covers B, and such that $\alpha \cong \beta' = \{U \mid U \in \beta, U \frown A \neq \phi\}$ where $\alpha \cong \beta'$ means that α is similar to β'. Then there exists a one-one correspondence i of the elements of α to the elements of β such that for any finite subcollection $\{U_{\mu m} \mid m=1,\cdots,k\}$ of α $\bigcap_{m=1}^{k} U_{\mu m} \neq \phi$ if and only if $\bigcap_{m=1}^{k} i(U_{\mu m}) \neq \phi$. In this paper we will use i in the case only when each element U_μ of α is contained in the element $i(U_\mu)$ of β. Now this i induces a homomorphism of $C^n(N(\alpha); G)$ into $C^n(N(\beta); G)$ where $C^n(N(\alpha); G)$ and $C^n(N(\beta); G)$ are n-cochain groups of $N(\alpha)$ and $N(\beta)$, respectively. We denote this homomorphism by $i(A, B)$. Since $\alpha \cong \beta'$, it is true that $i(A, B)(z^n)$ is an n-noncycle of $N(\beta')$ for each n-cocycle z^n of $N(\alpha)$. (n is a non-negative integer.)

Definition. We shall say that a space X has $D(X; G)=-1$ if and only if X is the empty set. For any non-negative integer n we shall say that X has $D(X; G) \leqq n$ if and only if, for any integer m such as $m \geqq n$ and for any closed set A of X, the homomorphism $H^m(X; G) \to H^m(A; G)$ induced by inclusion is onto.*)

We state now for reference several results to be used below.

Theorem 2.1. (*Exactness*) *If X is a space and if A is a closed set of X, then the following sequence is exact*

$$\cdots \to H^n(X; G) \to H^n(A; G) \to H^{n+1}(X, A; G) \to H^{n+1}(X; G) \to \cdots.$$

Theorem 2.2. (*Mayer-Vietoris*) *If X is a space and if X_1, X_2 are closed sets of X such as $X=X_1 \smile X_2$, then the following sequence is exact*

$$\cdots \to H^n(X; G) \to H^n(X_1; G) \times H^n(X_2; G) \to H^n(X_1 \frown X_2; G) \to H^{n+1}(X; G) \to \cdots.$$

The above theorems are found in [3] and [4; p. 43].

Theorem 2.3. (*Katetov*) *If X is a collectionwise normal Hausdorff space, then for each closed set S of X and for each locally finite open covering $\{U_\xi\}$ of S there exists a locally finite collection $\{V_\xi\}$ of open sets of X satisfying the following condition $C(S, \{U_\xi\}$,*

*) By a kind letter from Professor K. Morita we have learned that for compact spaces this definition coincides with Alexandroff's original definition.

$\{V_\xi\})$: $\bigcup_\xi V_\xi \supset S$, $V_\xi \frown S \subset U_\xi$ *and the correspondence* $V_\xi \leftrightarrow U_\xi$ *induces* $\{V_\xi\} | S \cong \{U_\xi\} \cong \{V_\xi\} \cong \{\bar{V}_\xi\}$ (cf. [5], Theorem 3.2).

3. Main theorems.

Theorem 3.1. (*Monotone*) *Let X be a space and let Y be a closed set of X. If $D(X; G) \leqq n$, then we have $D(Y; G) \leqq n$.*

This is an immediate consequence of the definition.

Theorem 3.2. (*Sum*) *Let X be a space and let $\{X_k\}$ be a countable collection of closed sets of X such as $\bigcup_{k=1}^{\infty} X_k = X$. If each X_k has $D(X_k; G) \leqq n$, then we have $D(X; G) \leqq n$.*

Proof. It is sufficient to show that for any integer $m (\geqq n)$ and for any closed set A of X the homomorphism $H^m(X; G) \to H^m(A; G)$ is onto. Hence for each element e of $H^m(A; G)$ we shall construct an m-cocycle z^m_ζ such as $\{z^m_\zeta | A\} = e$ where $\{z^m_\zeta\}$ means the cohomology class of $N(\zeta)$ containing z^m_ζ.

Since $D(X_1; G) \leqq n$, by Theorem 2.2 we get an element e_1 of $H^m(A \smile X_1; G)$ which is an extension of e. Quite similarly, we get an element e_2 of $H^m(A \smile X_1 \smile X_2; G)$ which is an extension of e_1.

Let α be a locally finite open covering of $A \smile X_1 \smile X_2$ such that there exists an m-cocycle z^m_α of $N(\alpha)$ which is a representation of e_2. By Theorem 2.3 there exists a locally finite collection β of open sets of X which satisfies the condition $C(A \smile X_1 \smile X_2, \alpha, \beta)$.

Since X is normal, there exist open sets P_2, P_1 and P_0 of X such that $A_2 = A \smile X_1 \smile X_2 \subset P_2 \subset \bar{P}_2 \subset \bigcup \{U | U \in \beta\}$, $A_1 = A \smile X_1 \subset P_1 \subset \bar{P}_1 \subset P_2$ and $A \subset P_0 \subset \bar{P}_0 \subset P_1$.

Let $\alpha_k = \beta | \bar{P}_k (k=0, 1, 2)$, $z^m_{\alpha_2} = i(A_2, \bar{P}_2) \Pi_{\alpha, \beta | A_2} z^m_\alpha$ and let $z^m_{\alpha_k} = z^m_{\alpha_2} | \bar{P}_k$ $(k=0, 1)$. From $\alpha \cong \alpha_2$ $z^m_{\alpha_2}$ is a cocycle of $N(\alpha_2)$ and, therefore, $z^m_{\alpha_k}$ is also a cocycle of $N(\alpha_k)$ $(k=0, 1)$. Furthermore, from $z^m_{\alpha_2} | A_2 = \Pi_{\alpha, \beta | A_2} z^m_\alpha$ we have $\{z^m_{\alpha_2} | A_2\} = \{z^m_\alpha\}$ on A_2 and, consequently, we have $\{z^m_{\alpha_2}\} | A = e_2 | A = e$.

To obtain z^m_ζ we shall construct m-cocycles $z^m_{\alpha_k} (k=2, 3, \cdots)$ by induction.

Let us suppose that we have constructed $a_k)$ $(2 \leqq k \leqq l)$ such that

$$a_k) \begin{cases} P_k: & \text{open in } X;\ P_k \supset A_k = \bar{P}_{k-1} \smile X_k, \\ \alpha_k: & \text{locally finite open covering of } \bar{P}_k;\ \alpha_k | \bar{P}_{k-2} = \alpha_{k-2} \\ z^m_{\alpha_k}: & m\text{-cocycle of } N(\alpha_k);\ z^m_{\alpha_k} | \bar{P}_{k-2} = z^m_{\alpha_{k-2}}. \end{cases}$$

We shall now show that there exist P_{l+1}, α_{l+1} and $z^m_{\alpha_{l+1}}$ satisfying the condition $a_{l+1})$.

Let $F = Fr(P_l) \smile X_{l+1}$ where $Fr(P_l)$ is the boundary of P_l in X.

Let γ be a locally finite open covering of $X_{l+1} - P_l$ such that there exists an m-cocycle z^m_γ of $N(\gamma)$ such as $\{z^m_\gamma\} | F = \{z^m_{\alpha_l} | F\}$. Since $D(X_{l+1}; G) \leqq n$, it is possible to obtain such z^m_γ by Theorem 2.2.

By Theorem 2.3 there exists a locally finite collection ε of open sets of A_{l+1} satisfying the conditions $C(F, \alpha_l|F, \varepsilon)$, $C(F, \gamma|F, \varepsilon)$ and having the property that for any element U of ε $U\frown\bar{P}_{l-1}=\phi$. And, furthermore, $z^m_{\varepsilon\alpha_l}\sim z^m_{\varepsilon\gamma}$ on $N(\varepsilon|F)$ where $z^m_{\varepsilon\alpha_l}=\Pi_{\alpha_l|F,\varepsilon|F}(z^m_{\alpha_l}|F)$ and $z_{\varepsilon\gamma}=\Pi_{\gamma|F,\varepsilon|F}(z^m_\gamma|F)$. So, there exists an $(m-1)$-cochain $c^{m-1}_\varepsilon(\in C^{m-1}(N(\varepsilon|F); G))$ such that $\nabla c^{m-1}_\varepsilon=z^m_{\varepsilon\alpha_l}-z^m_{\varepsilon\alpha}$ where ∇ is a coboundary operator.

Let $\beta=\alpha_l|P_l\smile\varepsilon\smile(\gamma|X_{l+1}-\bar{P}_l)$. Then β is a locally finite open covering of A_{l+1}. Since $\beta|F=\varepsilon|F\cong\varepsilon$, we have

$$\nabla c^{m-1}_\beta=z^m_{\beta 0}-z^m_{\beta 1}+c^m_\beta \tag{1}$$

where $c^{m-1}_\beta=i(F, A_{l+1})c^{m-1}_\varepsilon$, $z^m_{\beta 0}=i(F, A_{l+1})z^m_{\varepsilon\alpha_l}$, $z^m_{\beta 1}=i(F, A_{l+1})z^m_{\varepsilon\gamma}$ and c^m_β is an element of $C^m(N(\beta)-N(\varepsilon); G)$.

From the construction of β $N(\beta)-N(\varepsilon)$ is divided into two disjoint subcomplexes of $N(\beta)$. Accordingly, we have

$$c^m_\beta=c^m_{\beta 0}+c^m_{\beta 1} \tag{2}$$

where $c^m_{\beta 0}$ and $c^m_{\beta 1}$ are m-cochains of $N(\beta)$ such that $(c^m_{\beta 0})_0\frown(X_{l+1}-\bar{P}_l)=\phi$ and $(c^m_{\beta 1})_0\frown P_l=\phi$.

Let

$$z^m_\beta=z^m_{\gamma\beta}-z^m_{\gamma\beta}|F+z^m_{\alpha_l\beta}+c^m_{\beta 1}$$

where $z^m_{\alpha_l\beta}=\Pi_{\alpha_l,\beta|\bar{P}_l}z^m_{\alpha_l}$ and $z^m_{\gamma\beta}=\Pi_{\gamma,\beta|X_{l+1}-P_l}z^m_\gamma$.

Now we shall show that z^m_β is an m-cocycle of $N(\beta)$. It is sufficient to show that for any $(m+1)$-simplex t of $N(\beta)$ $KI[\nabla z^m_\beta\colon t]=0$ where KI denotes the Kronecker index. (Cf. [2]) For this purpose we shall distinguish three cases.

Case 1). $(t)_0\frown F\neq\phi$. In this case

$$KI[\nabla z^m_\beta\colon t]=KI[\nabla z^m_{\alpha_l\beta}\colon t]=0,$$

because $z^m_{\alpha_l\beta}|F$ is a cocycle of $N(\varepsilon)$.

Case 2). $(t)_0\subset\bar{P}_l-F$. Similarly as in the case 1) we have

$$KI[\nabla z^m_\beta\colon t]=KI[\nabla z^m_{\alpha_l\beta}\colon t]=0.$$

Case 3). $(t)_0\subset X_{l+1}-\bar{P}_l$. In this case using (1) and (2) we have

$$\begin{aligned}KI[\nabla z^m_\beta\colon t]&=KI[\nabla(z^m_{\gamma\beta}-z^m_{\gamma\beta}|F+z^m_{\alpha_l\beta})\colon t]+KI[\nabla c^m_{\beta 1}\colon t]\\&=KI[\nabla(z^m_{\gamma\beta}-z^m_{\gamma\beta}|F+z^m_{\alpha_l\beta})\colon t]+KI[\nabla(z^m_{\beta 1}-z^m_{\beta 0}-c^m_{\beta 0})\colon t]\\&=KI[\nabla z^m_{\gamma\beta}\colon t]+KI[\nabla z^m_{\alpha_l\beta}|F-z^m_{\beta 0})\colon t]+KI[\nabla(z^m_{\beta 1}-z^m_{\gamma\beta}|F)\colon t]\\&=0,\end{aligned}$$

because $z^m_{\gamma\beta}$ is a cocycle of $N(\beta|X_{l+1}-P_l)$ and $z^m_{\alpha_l\beta}|F=z^m_{\beta 0}$, $z^m_{\beta 1}=z^m_{\gamma\beta}|F$. Thus the proof that z^m_β is a cocycle of $N(\beta)$ is completed.

By Theorem 2.3 there exists a locally finite collection δ of open sets of X satisfying the condition $C(A_{l+1}, \beta|A_{l+1}-P_l), \delta)$ and satisfying the condition that for any element U of δ $U\frown\bar{P}_{l-1}=\phi$.

Since X is normal, there exists an open set P_{l+1} of X such that

$$\bigcup\{U|U\in\beta'\}\supset\bar{P}_{l+1}\supset P_{l+1}\supset A_{l+1}$$

where $\beta'=\beta|P_l\smile\delta$.

Let $\alpha_{l+1}=\beta'|\bar{P}_{l+1}$ and let $z^m_{\alpha_{l+1}}=i(A_{l+1}, \bar{P}_{l+1})\Pi_{\beta,\alpha_{l+1}|A_{l+1}}z^m_\beta$. Then from the construction of α_{l+1} and by the definition of $z^m_{\alpha_{l+1}}$ we have

$$a_{l+1})\begin{cases} P_{l+1}\text{: open in } X;\ P_{l+1}\supset A_{l+1}, \\ \alpha_{l+1}\text{: locally finite open covering of } \bar{P}_{l+1};\ \alpha_{l+1}|P_{l-1}=\alpha_{l-1}, \\ z^m_{l+1}\text{: } m\text{-cocycle of } N(\alpha_{l+1});\ z^m_{\alpha_{l+1}}|\bar{P}_{l-1}=z^m_{\alpha_{l-1}}. \end{cases}$$

Thus we can construct the open set P_k and m-cocycle $z^m_{\alpha k}$ satisfying the condition a_k) for $k=2, 3, \cdots$, by an inductive process.

Let us put $\zeta=\bigcup_{k=2}^{\infty}(\alpha_k|P_{k-2})$. (We identify all elements which are the same set.) Then ζ is a locally finite open covering of X, because the collection $\{P_k-\bar{P}_{k-2}|k=2, 3, \cdots\}\smile\{P_1\}$ is a star-finite open covering of X.

Finally, we put

$$z^m_\zeta=i(\bar{P}_0, X)(z^m_{\alpha_2}|\bar{P}_0)+[i(\bar{P}_1, X)(z^m_{\alpha_3}|\bar{P}_1)-i(\bar{P}_0, X)(z^m_{\alpha_3}|\bar{P}_0)] \\ +[i(\bar{P}_2, X)(z^m_{\alpha_4}|\bar{P}_2)-i(\bar{P}_1, X)(z^m_{\alpha_4}|\bar{P}_1)]+\cdots.$$

Then z^m_ζ is an m-cocycle of $N(\zeta)$. For, if z^m_ζ is not a cocycle, then we have $KI[\nabla z^m_\zeta: t]\neq 0$ for some $(m+1)$-simplex $t=(U_0,\cdots, U_{m+1})$ of $N(\zeta)$. But from the construction of ζ there exists P_k such that $U_l\subset P_k(l=0,\cdots, m+1)$. Therefore, $0\neq KI[\nabla z^m_\zeta: t]=KI[\nabla z^m_{\alpha_{k+2}}: t)$. This contradicts the assumption that $z^m_{\alpha_{k+2}}$ is a cocycle. Moreover, since $z^m_\zeta|A=i(\bar{P}_0, X)(z^m_{\alpha k}|\bar{P}_0)|A=z^m_{\alpha_2}|A$, we have $\{z^m_\zeta\}|A=e$. This completes the proof of Theorem 3.2.

Corollary 3.3. *Let X be a space and let $\{K_\lambda\}$ be a locally finite and star-countable collection of closed sets of X. If each K_λ has $D(K_\lambda; G)\leqq n$, then we have $D(\bigcup_\lambda K_\lambda; G)\leqq n$.*

Proof. If we decompose $\{K_\lambda\}$ into the collection $\{\mathfrak{F}_\mu\}$ of components (cf. [8]), then each $\mathfrak{F}_\mu$ is a countable subcollection of $\{K_\lambda\}$ and each $F_\mu=\bigcup\{K_\lambda|K_\lambda\in\mathfrak{F}_\mu\}$ is a closed set of X. By Theorem 3.2 we have $D(F_\mu; G)\leqq n$. Since $\{K_\lambda\}$ is a locally finite, the collection $\{F_\mu\}$ is discrete in X. Therefore, we have $D(\bigcup_\mu F_\mu; G)=D(\bigcup_\lambda K_\lambda; G)\leqq n$.

4. Relations between $D(X; G)$ and other dimension functions.

In this section we indicate by *dim* the Lebesque's covering dimension defined as follows: If for any locally finite open covering of X there exists an open refinement of order not greater than $n+1$, then we define $dim X\leqq n$. (Cf. [7].)

Theorem 4.1. *If X is a space, then dim $X\geqq D(X; G)$ is concluded.*

Proof. Let $dim\, X\leqq n$. Then we have $H^m(X, A; G)=0$ for arbitrary closed set A and for each $m>n$. Hence it is enough to show that $H^n(X; G)\rightarrow H^n(A; G)$ is onto.

Let $\{z^n_\alpha\}$ be an arbitrary element of $H^n(A; G)$ and let α be a locally finite open covering of A such that z^n_α is a cocycle of $N(\alpha)$.

Using Theorem 2.3 there exists a (locally finite) subcollection α of open covering β of X such that the order of β does not exceed $n+1$ and satisfying the condition $C(A, \alpha, \alpha')$. Then by the properties of α' $i(A, X)\Pi_{\alpha,\alpha'|A}z^n_\alpha$ is an extension of $\{z^n_\alpha\}$. This completes the proof.

In particular, *if dim X is finite, we have* $D(X; Z)=dim\, X$ *where* Z *is the group of all integers* (cf. [2], Corollary 3.6 and Theorem 5.2). Thus we have immediately the following:

Corollary 4.2. *If dim X is finite,* $D(X; G)\leqq D(X; Z)$ *is true for an arbitrary* G.

Theorem 4.3. *If* X *is a locally compact (paracompact Hausdorff) space, then we have* $cdX=D(X; G)$.

Proof. Since X is a locally compact, paracompact Hausdorff space, there exists a star-finite open covering $\{U_\xi\}$ of X such that $\{\overline{U}_\xi\}$ is star-finite and such that each $\overline{U}_\xi$ is compact (cf. [6]). Let $\{\mathfrak{F}_\varphi\}$ be the collection of components of $\{\overline{U}_\xi\}$ (cf. [8]); then each $\mathfrak{F}_\varphi$ is a countable subcollection of $\{\overline{U}_\xi\}$. Since $cdA=D(A; G)$ for an arbitrary compact set A of X, for each φ $cd[\bigcup\{\overline{U}_\xi|\overline{U}_\xi\in\mathfrak{F}_\varphi\}]=D(\bigcup\{\overline{U}_\xi|\overline{U}_\xi\in\mathfrak{F}_\varphi\}; G)$ by Sum Theorem. If we put $F_\varphi=\bigcup\{\overline{U}_\xi|\overline{U}_\xi\in\mathfrak{F}_\varphi\}$, then the collection $\{F_\varphi\}$ is discrete. Hence we have $cdX=D(X; G)$.

We indicate by $indX$ the inductive dimension of Urysohn and Menger.

Theorem 4.4. *If* X *is an S-space, then we have* $D(X; G)\leqq indX$.

Proof. By Theorem 4.1 we have inequality $D(X; G)\leqq dimX$. The inequality $dimX\leqq indX$ was established (cf. [7], Theorem 5.2). This shows that $D(X; G)\leqq indX$.

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