109. *Some Remarks on the Finitary Method*

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In this paper the present writer intends to make inquiries about the 'finitary' method employed by formalists on the basis of the results attained in his last attempt.[1]

1. The Problems Propounded. Formalism is undoubtedly a very strong standpoint in the foundations of mathematics, and the finitary method adopted by its adherents also is so effectual that its validity seems almost irrefutable. It is true that the method in question has been chosen as the purest and most unquestionable one in the course of a persistent pursuit of accuracy. But when we consider carefully why it has been called in, what concepts it mobilizes, and how it is applied to actual cases, we suspect that it is susceptible of some questioning. Again, when it is put to practical use in metamathematical speculation within the boundaries of formalism, there seems to be a subtle problem in the relations between the thinking and its object. The following pages are devoted to a discussion of these issues,[2] and not to an all-round and exhaustive study of the finitary method.

2. A Survey of Formalism as a Preliminary Step. As is well known, in formalism the foundamental concepts appearing in the mathematical system which it treats of and the hypotheses needed for the evolution of the system, including the 'Schluss-schemata', are symbolized and formalized in 'Zeichen ohne Zeigen', and actual reasoning is carried on from the finitary standpoint. This actual speculation is called 'metamathematics' or a 'Beweistheorie' in contradistinction to formalized mathematics. In metamathematics is allowed free use of mathematics based on the finitary standpoint, especially of the results attained by the finitary natural number theory. The finitary standpoint consists in a way of reasoning, independently of axiomatic hypotheses, by means of an 'inhaltliche Gedankenexperiment' which directly concerns itself with an object

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1) S. Nakazima: "Foundation" and Formalism, Proc. Japan Acad., **37**, 452–456 (1961).

2) The present writer thinks that opinions may be divided on the very significance of such a discussion and that the very possibility of a variety of views on this point is a proof of the peculiar position the foundations of mathematics holds among the branches of mathematics.

Cf. 1) and The Foundations of Mathematics and Philosophical Point of View.

actually present to the eyes. The above is a rough-hewn definition of formalism.[3]

3. The Finitary Natural Number and its Use in Metamathematics. From the finitary standpoint the natural number is interpreted to be a figure to be defined recursively as follows:

1) 1 is a natural number.

2) If n is a natural number, so is $n+1$.

3) A natural number is composed only of the above-mentioned elements: a figure from which no meaning can be drawn, and which has nothing but qualities perceived by sight alone. Hence comes the necessity of using the term 'figure' in addition to the term 'number'. But then it is inconceivable that a theory concerning one should be of any use in speculating on the other. Because a figure is an object, but cannot make a content of reasoning. It is true, however, that finitary natural numbers are used in metamathematics, but, in the majority of cases, they are natural numbers as denoting an 'Anzahl'. They are to be seen abundantly in the demonstrations of theorems in syntaxes of symbolic logic and formalized mathematics, especially in the demonstrations of theorems which have the nature of a universal judgement. But the natural number as denoting an 'Anzahl' is not one as a figure, but certainly one which is thought to be apprehended 'anschaulich-inhaltlich'. Then it cannot be denied that there is a confusion of thought about where the object exists. In defining a figure actually existing before our eyes which has been constructed recursively, we can surely tell how many figures such as 1 and + are contained in it. Therefore, the figure and the 'Anzahl' are related in some way, but they do not represent the same concept. It must however be added that the foregoing analysis of the relations between the figure and the 'Anzahl' is no strict and definitive one, in that the very concept of a figure present before the eyes cannot be said to be sufficiently clear (on this point further remarks shall be made later on), and then the 'Anzahl' also, as it has already been said, is no more than what we think we apprehend 'anschaulich-inhaltlich'. When we realize that the 'Anzahl' cannot be apprehended except by intuition, we seem to have come up against a wall in the course of our researches in mathematics. Anyway, neither of the two concepts cannot be said to be sufficiently 'clair et distinct' (Descartes).

4. Demonstration by the Recursive Method. In metamathematics mathematical induction is applied to finitary natural numbers. Here it has a different meaning from what it has in the classical natural number theory, although it is expressed in much the same

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3) This definition is too narrow. *Cf.* Ch. 1 of the second paper cited in 2).

form in both cases. So it is called the recursive or inductive method when it is employed in metamathematics. Because in the classical natural number theory mathematical induction makes assertions concerning *all* the natural numbers, while, when employed from the finitary standpoint, it only makes assertions concerning an *arbitrary* number of natural numbers. In the latter case, for instance, one reasons in the following way concerning a proposition the truth or otherwise of which may be judged factually and intuitively with reference to individual natural numbers; when one knows that it is true in regard to 1, and that, if it is true in regard to n, it is true in regard to $n+1$ as well, one can say that it holds good with an arbitrary number a. There one finds the basis of the validity of such an inference in the notion that, if one has any concrete natural number placed before one's eyes, one can proceed with an argument step by step until one has really arrived at a, because finitary natural numbers are constructed recursively. To speak exactly, however, the number here means the figure, and a closer examination of the above process of reasoning shows that mathematical induction can be used in this way only by unconsciously presupposing the concept of the 'Anzahl'. But no convincing analysis has as yet been made of the relations between mathematical induction thus employed and the 'Anzahl'. It seems, therefore, that the finitary standpoint cannot yet be said to be the 'purest and most unquestionable' one.

By the way, of the great cogency of this recursive method of demonstration, there can be seen any number of instances in treatises written on the lines laid down by formalism. It is to be noted too that this method displays its powerfulness in the way of demonstration by transferring an argument about an essentially infinite object to a region of the apparently finite, from which fact, perhaps, is derived the epithet 'finitary'. And it is evident that such transference is made possible be converting an affirmative judgement into a double negative.

5. The Necessity of Introducing the 'Irgendein'. In the chapters 2, 3 and 4 there often appeared the concept of 'an individual concrete object put before the eyes'.[4] But when we look into actual demonstrations in metamathematics, we find that we cannot take this 'concrete object' on trust as really such. It seems that the finitary method has originally been adopted with a view to conducting demonstrations by the verification of individual cases. A theorem, in essence, has the nature of a universal judgement in one way or other, whether it may be affirmative or negative. Whereas, if an

4) The present writer thinks that such an object as this is really an 'Irgendein'. So he will use this term hereafter. The reason why he thinks so will be seen later.

object about which we assert something is really an individual having a substantial content, our assertion will be a mere verification of the fact with no inference contained in it; in other words, it will leave no room for any theory whatever. (Indeed, this is admittedly a statement rather too sweeping, but perhaps not quite wide of the truth, roughly speaking.) On the other hand, a universal judgement cannot but be verified by a deduction from another proved one. (The inductive method shall be reserved for a later discussion.) But if this method is carried too far, it will be involved into a 'regressio ad infinitionem'. This is the reason why in the foundations of mathematics the inductive method is valued highly in a respect. In fine, even a statement made in metamathematics has a universal nature, and so the verification of an individual in a literal sense is impossible. Thus one is compelled to appeal to the recursive method and regard one's object as a 'concrete thing placed before one's eyes'.

6. The Ambiguity of the 'Irgendein'. The so-called 'concrete object', however, is not a real individual but an 'Irgendein', that is, a thing which we conceive in the mind. This fact shows itself not only in the realities of metamathematics,[5)] but also in the employment of the term 'Gedankenexperiment' in an attempt to authenticate the finitary standpoint. Because the term implies in itself that the object exists in our imagination, but never literally 'before our eyes'. To make a parody of Mr. Russell's dictum, the concept of the 'concrete' in this case is not 'pradikabel'. Be that as it may, is the concept of the 'Irgendein' a sufficiently substantial and definite one? It cannot be considered to be a 'purest and most unquestionable' one. It will be only the knowledge of an individual that can be styled as such. For instance, the clarity of the concepts of 1, 2 and 3 is out of question. The same may be said with $2+3=5$. Needless to say, this is true only within the scope of primitive arithmetic, which shall be put out of the present consideration.

At any rate, the ambiguity of the 'Irgendein' comes from the fact that it is situated just at the intersection of 'das Allgemeine' and 'das Individuum'.

7. Individuality *vs.* Universality: a Gordian Knot in Mathematics. This problem has something eternal about it. In antiquity, individuality *vs.* universality was materialized in the opposition between

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5) For instance, one often says from the finitary standpoint, "If the natural number a is actually given". But, firstly, the natural number a is not an individual. Secondly, it has an 'irgendeine natürliche Zahl' n among its components, which proves that a is not an 'Individuum'. Furthermore, the recursive definition of the finitary natural number seems in itself to be guilty of vicious circle.

Cf. Ch. 3, 2) of this paper.

Heraclitus and the Eleatic school and between Antisthenes and Plato, and in the Middle Ages, it constituted the theme of the 'Universalienstreit' between Nominalism and Realism (in epistemology). The present writer thinks that the standpoint which may be called Individualism is the most infallible one. So he regrets that it cannot apply to an infinite object, for, to quote from Weyl, "Die Mathematik ist die Wissenschaft vom Unendlichen." Even a finitary natural number implies an endless series of many others, as it may be inferred naturally from the way in which it is constructed. Therefore, an object as seen from the finitary standpoint is at once 'endlich' and 'unbeschränkt'. It is in Cantor's terms, a 'Veränderlich-Endliches' and 'Potential-Unendliches', and further corresponds to an 'Uneigentlich-Unendliches'. That is, as is well known, the finitary method postulates the 'infinite' implicitly: it is in its turn aimed at apprehending the infinite. Here is also a Gordian knot in the foundations of mathematics. Hence comes that certain ambiguity of the finitary standpoint. Dr. Suetuna,[6] who takes a quite different standpoint from formalism, in dealing with a similar problem, subdivides the 'Irgendein' into the 'Irgendein-Beliebig-Bestimmtes' and the 'Beliebig-Allgemeines' or the 'Konkret-Allgemeines' (im Hegelischen Sinne). This will be a noteworthy attempt at the infinite.

8. Epilogue. In the foregoing dissertation which has been rather abstract and conceptual throughout, the present writer's constant wish has been to investigate the 'ambiguity', so to speak, of the natural number used as a tool in metamathematics. He thinks that the establishment of a convincing natural number theory with a substantial content is the first step towards laying foundations for mathematics. There are already a good many natural number theories at present, indeed, but it will not be futile to seek to set up a more satisfactory one. To the accomplishment of this task choice between formalism and intuitivism will be of little importance.

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6) Z. Suetuna: Ueber die Grundlagen der Mathematik. I, J. Math. Soc. Japan, **3**, 59-68 (1951); II, Proc. Japan Acad., **27**, 389-392 (1951); III, Proc. Japan Acad., **29**. 91-95 (1953).

Z. Suetuna: Ueber den Begriff der Totalität in der Mathematik, Ann. Japan Association for Philosophy of Science, **1**(1), 33-40 (1956).