105. *Relations among Topologies on Riemann Surfaces. IV*

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Example 4. Let $\mathfrak{R}$ be a circle $|z+1|<1$. Let R_n be a domain such that R_n: $\frac{1}{2^n}\geqq|z|\geqq\frac{1}{2^{n+1}}$, $|\arg z|\leqq\frac{\pi}{16}$ and put $\sum_{n=1}^{\infty}R_n=R$ and $D=\mathfrak{R}-R$. Domain $\mathfrak{D}$. Let Λ_n and Γ_n be domains as follows:

$$\Lambda_n: \frac{1}{2^{n+1}}+a_n>|z|>\frac{1}{2^n} \text{ and } a_n<\frac{1}{3\times 2^{n+1}},\ |\arg z|<\frac{\pi}{16},$$

$$\Gamma_n: \frac{1}{2}\left(\frac{1}{2^n}+\frac{1}{2^{n+1}}\right)\geqq|z|\geqq\frac{1}{2}\left(\frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}\right),\ |\arg z|\leqq\frac{\pi}{8},$$

where a_n will be determined. Then $\Gamma_n\supset\Lambda_n$ and dist $(\partial\Gamma_n, \Lambda_n)>0$. Let $G(z, p_0, \mathfrak{R})$ be the Green's function of $\mathfrak{R}$, where $p_0=-\frac{3}{2}$. Put $M_n=\max G(z, p_0, \mathfrak{R})$ on $\partial R_n+\partial R_{n+1}$. Let $w(z, \Lambda_n, D)$ be the harmonic measure of Λ_n-D relative to D. Now D is simply connected and dist $(\partial\Gamma_n, \Lambda_n)>0$. Hence by Lemma 3 or 5 we can find a constant a_n such that

$$M_n w(z, \Lambda_n, D)\leqq\frac{1}{4^n}G(z, p_0, D) \text{ on } \partial\Gamma_n. \tag{15}$$

We suppose a_n is defined as above. Put $\mathfrak{D}=\mathfrak{R}-R+\sum_{n=1}^{\infty}\Lambda_n$. Now $M_n w(z, \Lambda_n, D)=0=\frac{1}{4^n}G(z, p_0, D)$ on $\partial D-\Gamma_n$. Hence by the maximum principle $M_n w(z, \Lambda_n, D)\leqq\frac{1}{4^n}G(z, p_0, D)$ in $D-\Gamma_n$. By $M_n\geqq G(z, p_0, \mathfrak{R})$ $\geqq G(z, p_0, \mathfrak{D})$ on $\partial\Lambda_n$ we have $M_n\geqq M_n w(z, \Lambda_n, D)+G(z, p_0, D)\geqq G(z, p_0, \mathfrak{D})$

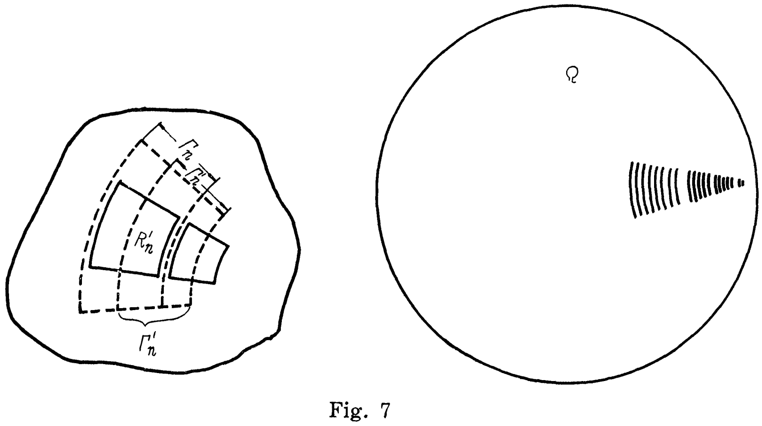

Fig. 7

on $\partial D\cap\partial\Lambda_n$. Now $M_n w(z, \Lambda_n, D)+G(z, p_0, D)=G(z, p_0, \mathfrak{D})=0$ on $\partial D-\partial\Lambda_n$. Hence by the maximum principle $\sum_{n=1}^{\infty} M_n w(z, \Lambda_n, D)+G(z, p_0, D)\geqq G(z, p_0, \mathfrak{D})\geqq G(z, p_0, D)$ in D and by (14)

$$\left(1+\sum_{n=1}^{\infty}\frac{1}{4^n}\right)G(z, p_0, D)\geqq G(z, p_0, \mathfrak{D})\geqq G(z, p_0, D) \text{ in } D-\sum_{n=1}^{\infty}\Gamma_n. \quad (16)$$

Let $\{p_n^i\}$ ($i=1, 2,$ and $n=1, 2, 3,\cdots$) be a sequence such that $p_n^i: |z| =\frac{1}{2^n}$, $\arg z=\frac{\pi}{4}$ for $i=1$ and $-\frac{\pi}{4}$ for $i=2$. Clearly $\{p_n^1\}$ in D determines different K-Martin's point from that of $\{p_n^2\}$, i.e. $\lim_n K(z, \{p_n^1\}, D)$ and $\lim_n K(z, \{p_n^2\}, D)$ are linearly independent. Now $p_n^i\in D -\sum_{n=1}^{\infty}\Gamma_n$. Let $\{p_{n'}^i\}$ be a subsequence of $\{p_n^i\}$ such that $\{p_{n'}^i\}$ determine K-Martins point relative to $\mathfrak{D}$. Then by (16) and by Lemma 8 ${}_{ex}(\lim_{n'} K(z, \{p_{n'}^i\}, D)$ (from D to $\mathfrak{D}$ relative to $\{v_n\}$)$>\infty$. Where $v_n =E\left[z: |z|<\frac{1}{2^n}\right]$. Thus we have

Proposition 1. *There exist at least two K-Martin's points of* $\mathfrak{D}$ *on* $z=0$.

Domain Ω. Let Γ_n' and T_n ($n=1, 2, 3,\cdots$) be a domain and a system of circular slits: $T_n=\sum_i t_n^i$ in R_n as follows:

$$\Gamma_n': \frac{1}{2^n}+\frac{a_{n-1}}{2}\geqq|z|\geqq\frac{1}{2^{n+1}}+\frac{a_n}{2}, \quad |\arg z|\leqq\frac{\pi}{8}.$$

T_n is contained in $R_n'=R_n-\Lambda_n$ and

$$t_n^i: |z|=\frac{1}{2}-\left(\frac{1}{2^{n+1}}-a_n\right)\frac{(i-1)}{k}, \quad |\arg z|<\frac{\pi}{16}, \quad i=1, 2,\cdots k+1.$$

Since $\operatorname{dist}(\partial\Gamma_n', \partial\mathfrak{D})>0$, $\min_{z\in\partial\Gamma'_n} G(z, p_0, \mathfrak{D})>0$. Now $G^{T_n}(z, p_0, \mathfrak{R})\to G^{R'_n}(z, p_0, \mathfrak{R})$ uniformly on $\partial\Gamma_n'$ as $k(n)\to\infty$. Hence there exists a number $k(n)$ such that

$$G^{T_n}(z, p_0, \mathfrak{R})-G^{R_n'}(z, p_0, \mathfrak{R})\leqq\frac{1}{5^n}G(z, p_0, \mathfrak{D}) \text{ on } \partial\Gamma_n'. \quad (17)$$

We suppose T_n is defined for every n. Put $\Omega=\mathfrak{R}-\sum_{n=1}^{\infty}R_n'+\sum_{n=1}^{\infty}(R_n' -T_n)$. By $\mathfrak{R}\supset\Omega\supset\mathfrak{D}$ and by Lemma 4 and by (17) we have $\frac{1}{5^n}G(z, p_0, \Omega)\geqq\frac{1}{5^n}G(z, p_0, \mathfrak{D})\geqq G^{T_n}(z, p_0, \mathfrak{R})-G^{R_n'}(z, p_0, \mathfrak{R})\geqq G^{T_n}(z, p_0, \Omega)-G^{R_n'}(z, p_0, \Omega)$ on $\partial\Gamma_n'$. On the other hand, $\frac{1}{5^n}G(z, p_0, \Omega)=0=G^{T_n}(z, p_0, \Omega) -G^{R'_n}(z, p_0, \Omega)$ on $\partial\Omega-\Gamma_n'$. Hence by the maximum principle

$$G^{T_n}(z, p_0, \Omega)-G^{R'_n}(z, p_0, \Omega)\leqq\frac{1}{5^n}G(z, p_0, \Omega) \text{ in } \Omega-\Gamma'_n.$$

Next by $T_n\subset\partial\Omega$ $G^{T_n}(z, p_0, \Omega)=G^{\Sigma T_n}(z, p_0, \Omega)=G(z, p_0, \Omega)$ and $G^{\Sigma R'_n}(z, p_0, \Omega)=G(z, p_0, \Omega-\sum R'_n)=G(z, p_0, \mathfrak{D})$. Hence by Lemma 4, $G(z, p_0, \Omega)-G(z, p_0, \mathfrak{D})\leqq\sum(G^{T_n}(z, p_0, \Omega)-G^{R'_n}(z, p_0, \Omega))\leqq\sum\frac{1}{5^n}G(z, p_0, \Omega)$ in $\Omega-\sum\Gamma'_n$. Now $p^i_n\in\Omega-\sum\Gamma'_n$ and we have $G(p^i_n, p_0, \Omega)\leqq\frac{5}{4}G(p^i_n, p_0, \mathfrak{D})$. Hence ${}_{ex}(\lim_n K(z, p^i_n, \mathfrak{D})$ (from $\mathfrak{D}$ relative to Ω)$)<\infty$. Hence by Proposition 1 and by Lemma 8 we have

Proposion 2. *There exist at least two K-Martin's points of Ω on $z=0$.*

We show that there exists only one N-Martin's point of Ω on $z=0$. Let $\Omega'=\Omega-D_0$, $D_0=E\left[z:\left|z+\frac{1}{2}\right|<\frac{1}{4}\right]$. Consider $N(z, p)$ of Ω'. Let $U(z)$ be a harmonic function in a domain G_r, $G_r=E[z:|z|<r]$ such that $U(z)$ has minimal Dirichlet integral over $\Omega'\cap G_r$. Then $U(z)=\lim_n U_n(z)$, where $U_n(z)$ is a harmonic function in $\Omega'\cap G_r\cap C_n$ $\left(C_n=E\left[z:|z+1|<1-\frac{1}{n}\right]\right)$ such that $U_n(z)=U(z)$ on $\partial G_r\cap C_n\cap\Omega'$ and $\frac{\partial}{\partial n}U_n(z)=0$ on $(\partial\Omega'+\partial C_n)\cap G_r$. Hence by the maximum principle $\sup_{\partial G_r\cap\Omega'\cap C_n}U_n(z)\geqq\sup_{G_r\cap\Omega'\cap C_n}U_n(z)\geqq\inf_{G_r\cap\Omega'\cap C_n}U_n(z)\geqq\inf_{\partial G_r\cap\Omega'\cap C_n}U_n(z)$ and by letting $n\to\infty$ $\sup_{\partial G_r\cap\Omega'}U(z)\geqq\sup_{G_r\cap\Omega'}U(z)\geqq\inf_{G_r\cap\Omega'}U(z)\geqq\inf_{\partial G_r\cap\Omega'}U(z)$. Put $\Gamma_r=E[z:|z|=r]$ and $z=re^{i\theta}$ and $L(z)=\int_{\Gamma_r}\left|\frac{\partial}{\partial r}U(z)\right|rd\theta$. Then by

$$\int_{E(r,r_0)}\frac{1}{r}dr\to\infty \text{ as } r\to 0$$

and $\displaystyle\int_{E(r,r_0)}\frac{L(r)}{r}dr\leqq\iint_{E(r,r_0)}\left\{\left(\frac{\partial U(z)}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial}{\partial\theta}U(z)\right)^2\right\}dr\,d\theta\leqq D(U(z))<\infty$,
we see that there exists a sequence $r_1>r_2\cdots$ such that $\sup_{\Gamma_{r_i}}U(z)-\inf_{\Gamma_{r_i}}U(z)\leqq\int_{\Gamma_{r_i}}\left|\frac{\partial}{\partial r}U(z)\right|rd\theta=L(r_i)\to 0$ as $r_i\to 0$, where $E(r, r_0)=I(r, r_0)-\sum_{n=1}^{\infty}\Lambda_n$ and $I(r, r_0)$ is the interval $r_0>z>r$ on the real axis. Whence $\lim_{z\to 0}U(z)$ exists. Now $D(N(z, p))<\infty$ over $G_{r'}:r'<r(p^i_n)$ is finite for any p^i_n and $N(z, p)$ has minimal Dirichlet integral over $G_{r'}$ and $\lim_{z\to 0}N(z, p^i_n)$ exists. By $N(z, p^i_n)=N(p^i_n, z)$ we have $\lim_n N(z, p^1_n)=\lim_n N(z, p^2_n)$ for any z, whence $\lim_n N(z, p^1_n)=\lim_n N(z, p^2_n)$. Thus $\{p^1_n\}$

and $\{p_n^2\}$ determine the same N-Martin's point of Ω on $z=0$ and $KM.T \not< NM.T$ and we have by Examples 3 and 4 the following

Theorem 4. b). $KM.T \not\gtrless NM.T$.

Example 5. Let $C=E[z:|z|<1]$ and $F_n=E\left[z:\frac{1}{2^n}\leqq z\leqq\frac{1}{2^n}+a_n\right]$ on the real axis. We suppose that $\sum_{n=1}^{\infty}F_n$ is so thinly distributed that $z=0$ may be an irregular point for the Dirichlet problem of $\Omega=C-\sum F_n$. Then $\overline{\lim_{z\to 0}}\, G(z, p_0, \Omega)=\delta>0$. Let $\{p_n\}$ be a sequence tending to $z=0$ such that $\varliminf_n G(p_n, p_0, \Omega)\geqq\frac{\delta}{2}$. Choose a subsequence $\{p_{n'}^1\}$ of $\{p_n\}$ such that $G(z, p_{n'}^1, \Omega)$ converges to a harmonic function (which is clearly non constant) denoted by $G(z, \{p_{n'}^1\}, \Omega)$. Let $\gamma_{n'}$ be a curve connecting $F_{n'}$ with $p_{n'}^1$. Then since ∂F_n is regular, $G(z, p_0, \Omega)=0$ for $z\in F_n$. And we can find $p_{n'}^2$ on $\gamma_{n'}$ such that $\lim_{n'} G(p_{n'}^2, p_0, \Omega)=\frac{\delta}{4}$. Choose a subsequence $\{p_{n''}^2\}$ of $\{p_{n'}^2\}$ such that $G(z, p_{n''}^2, \Omega)$ converges to $G(z, \{p_{n''}^2\}, \Omega)$. Next choose a subsequence $\{p_{n'''}^i\}$ of $\{p_{n''}^i\}$ $(i=1, 2)$ such that $\{p_{n'''}^i\}$ tends to a boundary point p^i with respect to Green's metric. Then dist $(p^1, p^2)=\inf_L\int d|e^{-G(z, p_0, \Omega)-ih(z, p_0, \Omega)}|>e^{\delta/2}-e^{\delta/4}>0$, whence $p^1\neq p^2$ with respect to Green's metric, where L is a curve connecting p^1 with p^2 and $h(z, p_0, \Omega)$ is the conjugate of $G(z, p_0, \Omega)$. On the other hand, ${}_{ex}G(z, \{p_{n'''}^i\}, \Omega)$ (from Ω to C relative to v_n) $\leqq G(z, p_0, C)<\infty: p_0=z=0$. $v_n=E[z:|z|<1/n]$. Now there exists only one linearly independent positive harmonic function in $C-p_0$ vanishing on ∂C. Hence by (14) of Lemma 8 such functions $G(z, \{p_n^1\}, \Omega)$, $G(z, \{p_{n'}^2\}, \Omega)\cdots$ are linearly dependent. On the other hand, by $G(z, \{p_{n'''}^i\}, \Omega)>0$ $\lim_{n'''} K(z, p_{n'''}^i, \Omega)$ exists and is equal to a $G(z, \{p_{n'''}^i\}, \Omega)$. But $K(z, \{p_{n'''}^i\}, \Omega)=1$ at $z=p_0$, whence by the linearly dependency $K(z, \{p_{n'''}^1\}, \Omega)=K(z, \{p_{n'''}^2\}, \Omega)$. Hence $\{p_{n'''}^1\}$ and $\{p_{n'''}^2\}$ determine the same K-Martin's point relative to Ω. Thus $KM.T \not> G.T$.

Example 6. Let R_1 be a unit circle $:|z|<1$ with slits $S_n: Im\, z=0$, $\frac{1}{2^n}\leqq Re\, z\leqq\frac{1}{2^n}+a_n$. Let R_2 be the identical leaf to R_1. We choose a_n so small that $z=0$ may be an irregular point of the Dirichlet problem of $R_i'=R_i-\sum_{n=1}^{\infty}S_n$. Connect R_1' and R_2' crosswise on $\sum S_n$. Then we have a Riemann surface $\mathfrak{R}=R_1+R_2$ of infinite genus. Since $z=0$ is irregular, we can find a sequence $\{p_n^1\}$ in R_1' such that $0<G(z, \{p_n^1\}, R_1')=\lim_n G(z, p_n^1, R_1')$ and $\lim_n G(z, p_n^1, \mathfrak{R})=G(z, \{p_n^1\}, \mathfrak{R})$ exist and $G(z, \{p_n^1\}, R_1')=aK(z, \{p_n^1\}, R_1'): 0<a<\infty$. Clearly $G(z, \{p_n^1\}, R_1')<G(z, \{p_n^1\}, \mathfrak{R})<\infty$. Whence ${}_{ex}(K(z, \{p_n^1\}, R_1')$ (from R_1' to $\mathfrak{R}$))$<\infty$. Similarly we can find $\{p_n^2\}$ in R_2' such that ${}_{ex}(K(z,\{p_n^2\},R_2')$ (from R_2' to $\mathfrak{R}$)$)<\infty$. Hence by $R_1'\cap R_2'=0$ and by (14) and by Lemma 8 ${}_{ex}(K(z,\{p_n^1\},R_1'))$ and ${}_{ex}(K(z,\{p_n^2\},R_2'))$ are linearly independent. Thus there exists at least two K-Martin's point of $\mathfrak{R}$ on $z=0$. Consider $G(z,p_0,\mathfrak{R}):p_0=1/2$. Then since p_0 is a branch point, $G(z,p_0,\mathfrak{R})=1/2\log\left|\dfrac{1-\dfrac{z^*}{2}}{z^*-\dfrac{1}{2}}\right|:z^*$ is the projection of z and $G(z,p_0,\mathfrak{R})$ is regular with respect to z^* in a neighbourhood of $z=0$. Hence $\int_L d|e^{-G(z,p_0,\mathfrak{R})-ih(z,p_0,\mathfrak{R})}|\to 0$ as the length of a curve $L\to 0$. Hence $\{p_n^1\}$ and $\{p_n^2\}$ determine the same point with respect to Green's metric. Hence $KM.T \not\prec G.T.$ Thus by Examples 5 and 6 $KM.T \not\asymp G.T.$

We show $NM.T\not\asymp G.T.$ In Example 5 suppose $\sum_{n=1}^{\infty}F_n$ is so thinly distributed on the real axis as $z=0$ is irregular and further $\int_{C\sum F_n} d\log r=\infty$ (in reality the irregularity of $z=0$ implies $\int_{C\sum F_n} d\log r=\infty$), where $C\sum F_n$ means the complementary set of $\sum F_n$ of the segment: $Im\, z=0$, $0<Re\, z<1$. Let $U(z)$ be a Dirichlet bounded and $U(z)$ has minimal Dirichlet integral in a neighbourhood $v_{r_0}=E[z:|z|<r_0]$ of $z=0$. Put $L(r)=\int_{\Gamma r}\left|\frac{\partial}{\partial n}U(z)\right|ds:\Gamma_r=E[z:|z|=r]$. Then there exists a sequence $\{r_i\}$ in $C\sum F_n$ such that $L(r_i)\to 0$ as $i\to\infty$. Whence as in Example 4, $\lim N(z,p_0)$ exists, where $N(z,p_0)$ is an N-Green's function of $C-\sum_{n=1}^{\infty}F-D_0$ and D_0 is a compact set of $C-\sum_{n-1}^{\infty}F_n$. Hence there Exists only one N-Martin's point on $z=0$ and $NM.T \succ\!\!\!\!+ G.T.$ We use example 6. Let R_1' and R_2' be the leaves of Example 6. Let $D=E\left[z:\left|z+\frac{1}{2}\right|<\frac{1}{4}\right]$ and put $R_i'=R_i-D_0$ and $\mathfrak{R}'=R_1'+R_2'$. Let $\tilde{\mathfrak{R}}'$ be the mirror image of $\mathfrak{R}'$ with respect to $|z|=1$. Connect $\mathfrak{R}'$ and $\tilde{\mathfrak{R}}'$ on $|z|=1$. Then we have a Riemann surface $\hat{\mathfrak{R}}$. Clearly $N(z,p,\mathfrak{R}')=G(z,p,\hat{\mathfrak{R}})+G(z,\tilde{p},\hat{\mathfrak{R}})$, where $\tilde{p}$ is the mirror image of p. Hence by the existence of linearly independent functions $G(z,\{p_n'\},\hat{\mathfrak{R}})$, $G(z,\{p_n^2\},\hat{\mathfrak{R}})$ we see there exist at least two linearly independent functions $N(z,\{p_n^1\},\mathfrak{R}')$ and $N(z,\{p_n^2\},\mathfrak{R}')$. Thus there exist at least two N-Martin's point on $z=0$ and $NM.T \not\prec G.T.$ Thus we have

Theorem 4. c). *$KM.T\not\asymp G.T$ and $NM.T\not\asymp G.T.$*