95. Some Characterizations of Fourier Transforms. IV

By Koziro IWASAKI
Musashi Institute of Technology, Tokyo
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1. Several years ago we proved

Theorem A. *Let a continuous even function $k(x)$ be the second derivative of a bounded function, and*

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}k(nt)\varphi(t)dt=\sum_{n=-\infty}^{\infty}\varphi(n) \tag{1}$$

for every function φ with compact support of class C^∞. Then

$k(x)=\cos 2\pi x$. (See [3].)

In what follow we shall give another proof of this theorem calculating the kernel function $k(x)$ explicitly.

2. If we apply the formula (1) to $\varphi^u(x)=\varphi(xu)$ with $u\neq 0$ we get

$$\frac{1}{|u|}\sum_{n=-\infty}^{\infty}\psi\left(\frac{n}{u}\right)=\sum_{u=-\infty}^{\infty}\varphi(nu), \tag{2}$$

where $\psi(x)=\int_{-\infty}^{\infty}k(xt)\varphi(t)dt$, or

$$\frac{1}{2|u|}\psi(0)+\sum_{n=1}^{\infty}\frac{1}{|u|}\psi\left(\frac{n}{u}\right)=\frac{1}{2}\varphi(0)+\sum_{n=1}^{\infty}\varphi(nu). \tag{3}$$

Because the function $\varphi(x)$ is a function with compact support and of class C^∞, $\sum_{n=1}^{\infty}\varphi(nu)$ is defined and of class C^∞ for any $u\neq 0$, and the support of this function is also compact.

On the other hand we have

$$\left|\int_{-\infty}^{\infty}k(xt)\varphi(t)dt\right|\leq c\cdot\frac{1}{x^2}\int_{-\infty}^{\infty}|\varphi''(t)|\,dt$$

using the hypothesis on $k(x)$, therefore

$$\sum_{n=1}^{\infty}\psi(nu)=O\left(\frac{1}{u^2}\right)$$

as u tends to infinity.

Now we shall calculate the Mellin-transform of the function $\sum_{n=1}^{\infty}\varphi(nu)$. Formally we get

$$\int_0^{\infty}u^{s-1}\sum_{n=1}^{\infty}\varphi(nu)du=\zeta(s)\Phi(s), \tag{4}$$

where $\zeta(s)$ is the Riemann zeta-function and $\Phi(s)$ is the Mellin transform of $\varphi(x)$. If we use the formula (3) we can transform the left hand side of (4) as follows:

$$\int_0^\infty=\int_0^1+\int_1^\infty$$
$$=\int_0^1 u^{s-1}\left(\frac{\psi(0)}{2u}-\frac{\varphi(0)}{2}\right)du+\int_0^1 u^{s-2}\sum_{n=1}^\infty\psi\left(\frac{n}{u}\right)du+\int_0^1 u^{s-1}\sum_{n=1}^\infty\varphi(nu)du$$
$$=\frac{\psi(0)}{2(s-1)}-\frac{\varphi(0)}{2s}+\int_1^\infty u^{-s}\sum_{n=1}^\infty\psi(nu)du+\int_1^\infty u^{s-1}\sum_{n=1}^\infty\varphi(nu)du.$$

Thus the Mellin-transform of $\sum_{n=1}^\infty\varphi(nu)$ exists for $Re\, s>1$. Similarly $\int_0^\infty u^{-s}\sum_{n=1}^\infty\psi(nu)du$ is equal to the last term of the above equations for $Re\, s>1$.

Because $\sum_{n=1}^\infty\varphi(nu)$ and $\sum_{n=1}^\infty\psi(nu)$ are uniformly convergent for $u\geq 1$ and $O(u^{-2})$, the last term of above equations is holomorphic for $Re\, s>-1$ with possibly exceptional points 0 and 1. Therefore (4) and

$$(5)\qquad \int_0^\infty u^{-s}\sum_{n=1}^\infty\psi(nu)du=\zeta(1-s)\Psi(1-s)$$

have the meaning for $Re\, s>-1$, where $\Psi(s)$ is the Mellin-transform of $\psi(x)$. But by the definition of $\psi(x)$ we have

$$\Psi(s)=\int_0^\infty x^{s-1}\psi(x)dx$$
$$=\int_0^\infty x^{s-1}\left(\int_{-\infty}^\infty k(xt)\varphi(t)dt\right)dx=\int_0^\infty x^{s-1}\left(2\int_0^\infty k(xt)\,\varphi(t)dt\right)dx$$
$$=2\int_0^\infty\int_0^\infty(ut^{-1})^{s-1}k(u)\varphi(t)\frac{1}{t}du\,dt=2K(s)\Phi(1-s),$$

where $K(s)$ is the Mellin-transform of $k(x)$. Therefore we get

$$\zeta(s)\Phi(s)=2\zeta(1-s)K(1-s)\Phi(s)$$

and

$$K(s)=\frac{1}{2}\frac{\zeta(1-s)}{\zeta(s)}=(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s).$$

This means that

$$k(x)=\cos 2\pi x.$$

(See [2] p. 204 (7.12.1).)

3. Using the similar method we can prove

Theorem B. *Let $k(x)$ be a continuous even function such that*

$$\int_0^\infty k(xt)\exp(-t^2)dt=O(x^{-1-\varepsilon})$$

for some $\varepsilon>0$ as x tends to infinity, and

$$\sum_{n=-\infty}^\infty\int_{-\infty}^\infty k(nt)\exp(-t^2u^2)dt=\sum_{n=-\infty}^\infty\exp(-n^2u^2)$$

for any $u>0$. *Then*

$$k(x)=\cos 2\pi x.$$

References

[1] S. Bochner and K. Chandrasekharan: Fourier Transforms, Princeton (1949).
[2] E. C. Titchmarsh: Introduction to the Theory of Fourier Integrals, Oxford (1937).
[3] K. Iwasaki: Some characterizations of Fourier transforms, Proc. Japan Acad., **35** (8), 423-426 (1956).