94. *Dirac Space*

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As is well-known, Dirac [1] gave a very elegant foundation of quantum mechanics, which contained however some self-contradictory concepts from the mathematical point of view. J. von Neumann [5] gave another foundation of the theory based upon his spectral theory of self-adjoint operators of Hilbert space. However the whole spectral theory is in fact not necessary and what Dirac actually needs is only that the self-adjoint operators $t\cdot$ and $\frac{1}{i}\frac{d}{dt}$ could be put in diagonal forms. There is also another justification of "improper" functions introduced by Dirac to put these operators in the diagonal forms by means of the theory of distributions of L. Schwartz. But this theory is not adequate to interpret inner product used in Dirac's theory. The purpose of this paper is to show that we can interpret the theory of Dirac in a more natural and mathematically rigorous way.

Recently A. Robinson developed non standard analysis [6], which is an adequate non Archimedian extension of real number field, and in which he succeeded to define infinite and infinitesimal and to develop rigorously infinitesimal calculus of Leibniz and Euler.

In this paper, we shall define Dirac space by an ultraproduct of $L_2(-\infty, \infty)$ and justify Dirac's method by using Robinson's consideration of infinite and infinitesimal.

Let I be the set of all positive integers.

A family $\mathfrak{F}$ of non-empty subsets of I is called a filter over I if

(i) $F_1 \in \mathfrak{F}$ and $F_2 \in \mathfrak{F}$ imply $F_1 \frown F_2 \in \mathfrak{F}$, and

(ii) $F_1 \in \mathfrak{F}$ and $F_1 \subseteq F_2 \subseteq I$ imply $F_2 \in \mathfrak{F}$.

A filter over I is an ultrafilter if it is maximal among the class of filters over I. It is easily proved that a necessary and sufficient condition that $\mathfrak{F}$ be an ultrafilter over I is that for $A \subseteq I$, $A \in \mathfrak{F}$ if and only if $I-A \notin \mathfrak{F}$. Hereafter we shall fix an ultrafilter $\mathfrak{F}_0$ over I which does not contain any finite subsets of I.

Now let R_0 be the set of all real numbers. Let a and b be elements of R_0^I. We use the notation $a=(a_1, a_2, \cdots)$ and $b=(b_1, b_2, \cdots)$ as usual, where a_i and b_i are real numbers and called the i-th coordinates of a and b respectively. $a \equiv b$, $a<b$, $a+b$ and $a \cdot b$ are defined to be $\{i \mid a_i = b_i\} \in \mathfrak{F}_0$, $\{i \mid a_i < b_i\} \in \mathfrak{F}_0$, $(a_1+b_1, a_2+b_2, \cdots)$ and $(a_1 \cdot b_1, a_2 \cdot b_2, \cdots)$ respectively. $\equiv$ is a congruence relation compatible with $<$, $+$, and $\cdot$.

R^* is defined to be $R_0{}^I/\equiv$ which is also written as $R_0{}^I/\mathfrak{F}_0$. R^* is an ordered field. This follows from the well developed general theory of ultraproduct ([2], [3], [4]) and is also proved directly. By the imbedding isomorphism j from R_0 to R^* defined by $j(a_0)=(a_0, a_0, a_0, \cdots)$, we can consider $R_0 \subseteq R^*$. R^* contains an infinite e.g. $(1, 2, 3, \cdots)$ which is larger than all elements of R_0.

Let M_1 be the set of all $a \in R^*$ such that $|a| < r$ for all positive $r \in R_0$. The elements of M_1 will be said to be infinitesimal. We define $a \doteqdot b$ if and only if $a-b$ is infinitesimal.

We see easily the following lemma.

LEMMA 1. *If a sequence $a_1, a_2, \cdots$ is convergent to a_0 in R_0, then $a \doteqdot a_0$ where $a=(a_1, a_2, \cdots)$.*

Now let C_0 be the set of all complex numbers. $C^*=C_0{}^I/\mathfrak{F}_0$ is defined in the same way as above. C^* can be considered as $R^* \oplus iR^*$. $\bar{\alpha}$ and $|\alpha|$ is defined to be $(\bar{\alpha}_1, \bar{\alpha}_2, \cdots)$ and $(|\alpha_1|, |\alpha_2|, \cdots)$ for $\alpha \in C^*$ where $\alpha=(\alpha_1, \alpha_2, \cdots)$. $\alpha \doteqdot \beta$ is defined to be $|\alpha-\beta| \doteqdot 0$.

Finally Dirac space $\mathfrak{D}$ is defined as follows: Let x and y be elements of $(L_2(-\infty, \infty))^I$. We use the notation $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ as usual, where x_i and y_i are elements of $L_2(-\infty, \infty)$ and sometimes written as $x_i(t)$ and $y_i(t)$ respectively. Let $\alpha =(\alpha_1, \alpha_2, \cdots)$ be an element of C^*. $x \equiv y$, αx, (x, y) and $x+y$ are defined to be $\{i \mid x_i - y_i\} \in \mathfrak{F}_0$, $(\alpha_1 x_1, \alpha_2 x_2, \cdots)$, $((x_1, y_1), (x_2, y_2), \cdots)$ and $(x_1+y_1, x_2+y_2, \cdots)$ respectively.

It is easily proved that $\equiv$ is a congruence relation compatible with these operations $\cdot$, $(\ ,\ )$ and $+$. $\mathfrak{D}$ is defined to be $(L_2(-\infty, \infty))^I/\equiv$. The injection map from $L_2(-\infty, \infty)$ into $\mathfrak{D}$ is defined by $x_0 \rightarrow (x_0, x_0, x_0, \cdots)$.

$\mathfrak{D}$ is a linear space over C^*. Also the following propositions hold for each $x, y, x_1, x_2 \in \mathfrak{D}$ and $\alpha \in C^*$.

1) $(x, x) \geq 0$ and $(x, x)=0$ is equivalent to $x=0$.

2) $(x, y)=\overline{(y, x)}$

3) $(x_1+x_2, y)=(x_1, y)+(x_2, y)$ and $(\alpha x, y)=\alpha(x, y)$.

These follow from the theory of ultraproduct ([2], [3], [4]), and are also proved directly.

$||x||$ is defined to be $\sqrt{(x, x)}$. $||x||=(||x_1||, ||x_2||, \cdots)$. $x \doteqdot y$ is defined to be $||x-y|| \doteqdot 0$.

LEMMA 2. *If $x_1, x_2, \cdots \rightarrow x_0$ in $L_2(-\infty, \infty)$, then $x \doteqdot x_0$ where $x=(x_1, x_2, \cdots)$.*

PROOF. This follows from Lemma 1 using

$||x-x_0||=(||x_0-x_1||, ||x_0-x_2||, \cdots)$.

$\delta_n(t)$ is defined by $\delta_n(t)=n+n^2t$, $\left(\text{if} -\frac{1}{n} < t \leq 0\right)$; $=n-n^2t$, (if

$0<t\leq\frac{1}{n}\Big)$; $=0$, (otherwise). δ is defined to be $(\delta_1, \delta_2, \cdots)$.

We have the following propositions.

PROPOSITION 1. $t\cdot\delta(t)\doteqdot 0$.

PROOF. This is proved by Lemma 1 and calculation of $||t\cdot\delta_n(t)||$.

PROPOSITION 2. $t\cdot\delta(t-a)\doteqdot a\delta(t-a)$.

PROOF. This follows directly from Proposition 1.

PROPOSITION 3. *For* $x\in L_2(-\infty, \infty)$, $x(t)\doteqdot\int_{-\infty}^{\infty}x(s)\delta(t-s)ds$

PROOF. As usual we define $(x*y)(t)$ to be $\int_{-\infty}^{\infty}x(t-s)y(s)ds$. $x*y=y*x$ and $||x*y||_2\leq||x||_2\cdot||y||_1$ are familiar.

Since the family C of continuous functions with compact carrier is dense in $L_2(-\infty, \infty)$, we have only to prove $x*\delta=x$ for $x\in C$. To prove this, it is sufficient to prove that $x*\delta_n$ are uniformly convergent to x, because the carriers of all $x*\delta_n$'s and x are contained in a closed interval. This is easily proved as follows.

$$\left|\int_{-\infty}^{\infty}x(t-s)\delta_n(s)ds-x(t)\right|=\left|\int_{-\infty}^{\infty}(x(t-s)-x(t))\delta_n(s)ds\right|$$
$$=\left|\int_{-\frac{1}{n}}^{\frac{1}{n}}(x(t-s)-x(t))\delta_n(s)ds\right|\leq\sup_{-\frac{1}{n}\leq s\leq\frac{1}{n}}|x(t-s)-x(t)|.$$

PROPOSITION 4. $(\delta(t-a), \delta(t-b))=0$ if $a\neq b$.

PROOF. Because $(\delta_n(t-a), \delta_n(t-b))=0$ if $|a-b|>\frac{2}{n}$.

REMARK. (δ, δ) is infinite.

Propositions 2–4 show that the self-adjoint operator $t\cdot$ can be put in the diagonal form.

Let U be a unitary operator of $L_2(-\infty, \infty)$, U can be also considered as a unitary operator of $\mathfrak{D}$ by defining Ux by $(Ux_1, Ux_2, \cdots)$. Clearly $(Ux, Uy)=(x, y)$.

Fourier transforms $\mathfrak{F}$ and $\bar{\mathfrak{F}}$ are defined by

$$(\mathfrak{F}x)(t)=\underset{n\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-n}^{n}e^{-ist}x(s)ds$$
$$(\bar{\mathfrak{F}}x)(t)=\underset{n\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-n}^{n}e^{ist}x(s)ds.$$

It is familiar that $\mathfrak{F}$ and $\bar{\mathfrak{F}}$ are unitary and $\frac{1}{i}\frac{d}{dt}=\bar{\mathfrak{F}}t\cdot\mathfrak{F}$.

$\eta_n^a(t)$ and $\eta^a(t)$ are defined to be $\bar{\mathfrak{F}}\delta_n(t-a)$ and $\bar{\mathfrak{F}}\delta(t-a)$ respectively.

PROPOSITION 5. $\frac{1}{i}\frac{d}{dt}\eta^a(t)\doteqdot a\eta^a(t)$.

PROOF. $\frac{1}{i}\frac{d}{dt}\eta^a(t)=\bar{\mathfrak{F}}t\cdot\mathfrak{F}\bar{\mathfrak{F}}\delta(t-a)=\bar{\mathfrak{F}}t\cdot\delta(t-a)\doteqdot\bar{\mathfrak{F}}a\delta(t-a)$

$$=a\bar{\mathfrak{F}}\delta(t-a)=a\eta^a(t).$$

PROPOSITION 6. *$(\eta^a(t), \eta^b(t))=0$ if $a\neq b$.*

PROOF. $(\eta^a(t), \eta^b(t))=(\bar{\mathfrak{F}}\delta(t-a), \bar{\mathfrak{F}}\delta(t-b))=(\delta(t-a), \delta(t-b))$.

PROPOSITION 7. $x(t)\doteqdot\int_{-\infty}^{\infty}(\mathfrak{F}x)(s)\eta^s(t)ds$.

PROOF. $\int_{-\infty}^{\infty}(\mathfrak{F}x)(s)\eta^s(t)ds=\int_{-\infty}^{\infty}(\mathfrak{F}x)(s)\bar{\mathfrak{F}}\delta(t-s)ds=((\mathfrak{F}x)(s), \mathfrak{F}\delta(t-s))_s$

$$=(x(s), \delta(t-s))_s=x*\delta\doteqdot x.$$

Propositions 5–7 show that the self-adjoint operator $\frac{1}{i}\frac{d}{dt}$ can be put in diagonal form.

PROPOSITION 8. *If $z\in L_2(-\infty, \infty)$ and $z(t)$ is continuous, then $(z, \delta)\doteqdot z(0)$.*

PROOF. $\left|\int_{-\infty}^{\infty}z(t)\delta_n(t)dt-z(0)\right|=\left|\int_{-\infty}^{\infty}(z(t)-z(0))\delta_n(t)dt\right|$

$$\leq\sup_{-\frac{1}{n}\leq t\leq\frac{1}{n}}|z(t)-z(0)|.$$

PROPOSITION 9. $(\delta(t-a), \eta^b(t))\doteqdot\frac{e^{-iab}}{\sqrt{2\pi}}$.

PROOF. $(\delta_n(t-a), \eta_n^b(t))=(\delta_n(t-a), \bar{\mathfrak{F}}\delta_n(t-b))$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-ist}\delta_n(s-b)\delta_n(t-a)dsdt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(s+b)(t+a)}\delta_n(t)\delta_n(s)dsdt$$

$$=\frac{e^{-iab}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i((s+b)(t+a)-ab)}\delta_n(t)\delta_n(s)dsdt$$

$$\therefore\ \left|(\delta_n(t-a), \eta_n^b(t))-\frac{e^{-iab}}{\sqrt{2\pi}}\right|$$

$$\leq\frac{1}{\sqrt{2\pi}}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{-i((s+b)(t+a)-ab)}-1)\delta_n(t)\delta_n(s)dsdt\right|$$

$$\leq\sup_{-\frac{1}{n}\leq t\leq\frac{1}{n},\ -\frac{1}{n}\leq t\leq\frac{1}{n}}|e^{-i((s+b)(t+a)-ab)}-1|.$$

We write $xy\overset{w}{=}$ if and only if $(x, z)\doteqdot(y, z)$ for all $z\in L_2(-\infty, \infty)$. The following propositions are proved in the familiar way.

PROPOSITION 10. $\delta(at)\overset{w}{=}\frac{1}{a}\delta(t)$ if $a>0$.

PROPOSITION 11. $\delta(t^2-a^2)\overset{w}{=}\frac{1}{2a}(\delta(t-a)+\delta(t+a))$ if $a>0$.

PROPOSITION 12. $\delta^*\delta\overset{w}{=}\delta$.

PROOF. We have only to prove $(\delta*\delta, x)\doteqdot\bar{x}(0)$ for $x\in C$. This is proved as follows:

$$\left|\int_{-\infty}^{\infty}(\delta_n*\delta_n)(t)\bar{x}(t)dt-\bar{x}(0)\right|=\left|\int_{-\infty}^{\infty}(\delta_n*\delta_n)(t)(\bar{x}(t)-\bar{x}(0))dt\right|$$
$$\leq \sup_{-\frac{2}{n}\leq t\leq \frac{2}{n}}|\bar{x}(t)-\bar{x}(0)|.$$

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