93. A Note on Metric General Connections

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In this note, the author will use the notations in [8], [9], [10], [11], [12]. He proved in [9] the following

Theorem A. *Let* $P=P^j_i\partial u_j\otimes du^i$ *and* $G=g_{ij}du^i\otimes du^j$ *be a normal tensor and a non-singular symmetric tensor on an n-dimensional differentiable manifold* $\mathfrak{X}$ *such that* P *is orthogonally related with* G. *Then, there exists a normal general connection* Γ *which satisfies the following conditions*:

(i) $P=\lambda(\Gamma)$, (ii) Γ *is proper, and*

(iii) Γ *is metric with respect to* G.

Furthermore, if we add to them the condition:

(iv) $S_k{}^j{}_h A^k_i=\frac{1}{2}A^j_l(P^l_{k;h}-P^l_{h;k})A^k_i$,

where A^j_i *are the local components of* A, $S_i{}^j{}_h=\frac{1}{2}(\Gamma^j_{ih}-\Gamma^j_{ih})$ *and the semi-colon ";" denotes the covariant derivatives with respect to Levi-Civita's connection made by* G, *then* Γ *is uniquely determined.*

In this theorem, A is the projection of $T(\mathfrak{X})$ onto the image of P with respect to the direct sum decomposition of $T(\mathfrak{X})$ by means of the image and the kernel of P.

On the other hand, we say a curve $C: u^j=u^j(t)$ in a space $\mathfrak{X}$ with a normal general connection $\Gamma=\partial u_j\otimes(P^j_i d^2u^i+\Gamma^j_{ih}du^i\otimes du^h)$ *is basic*, if its tangent vector at each point is invariant under A. In [12], he proved that *if* Γ *is contravariantly proper, that is*

$$N^j_k\Gamma^k_{lp}A^l_iA^p_h=0,$$

where $N^j_i=\delta^j_i-A^j_i$, *then we can uniquely parallel translate any A-invariant*[1] *contravariant vector at a point along a basic curve through the point, preserving the A-invariant property and if* Γ *is covariantly proper, that is*

$$A^j_k\Lambda^k_{lp}N^l_iA^p_h=0,$$

where $\Lambda^j_{ih}=\Gamma^j_{ih}-\partial P^j_i/\partial u^h$, *then the same fact holds good for covariant vectors.*

In [9], a normal general connection Γ was said *proper*, if $N\Gamma=0$,[2] that is

1) We say vectors or tensors are A-invariant, if they are invariant under the homomorphism A of $T(\mathfrak{X})$.

2) See [11], §1.

$$N^j_k\Gamma^k_{ih}=0,$$

hence it is contravariantly proper.

The author will show that for the uniquely determined general connection Γ in Theorem A we have $\Gamma N=0$, that is

$$A^j_{kh}N^k_i=0,$$

accordingly it is covariantly proper. And so he give a new the following

Theorem B. *Let $P=P^j_i\partial u_j\otimes du^i$ and $G=g_{ij}du^i\otimes du^j$ be a normal tensor and a non-singular symmetric tensor on an n-dimensional differentiable manifold $\mathfrak{X}$ such that P is orthogonally related with G, then there exists a uniquely determined normal general connection Γ such that $P=\lambda(\Gamma)$, metric with respect to G, $N\Gamma=0$, $\Gamma N=0$ and satisfies the generalized symmetric condition with respect to G:*

$$S_k{}^j{}_h A^k_i=\frac{1}{2}A^j_l(P^l_{k;h}-P^l_{h;k})A^k_i.$$

Proof. According to Theorem 1 in [9], a normal general connection Γ is uniquely determined under the above four conditions except $\Gamma N=0$ and it is given by

$$(1)\qquad \Gamma^j_{ih}=(\overline{[ih,l]}-\bar{S}_{lki}P_h{}^k-\bar{S}_{lkh}P^k_i)Q^l_pg^{pj}+\bar{S}_i{}^j{}_h,^{3)}$$

where $\overline{[ih,l]}$ are the Christoffel symbols of the first kind made by $\bar{g}_{ij}=g_{hk}P^h_iP^k_j$ and

$$\bar{S}_i{}^j{}_h=\frac{1}{2}A^j_l(P^l_{i;h}-P^l_{h;i}),\ \bar{S}_{lki}=g_{kj}\bar{S}_l{}^j{}_i.$$

In the first place, we notice that

$$(2)\qquad g^{jk}A^h_kg_{hi}=A^j_i \text{ and } g^{jk}N^h_kg_{hi}=N^j_i$$

which we can easily prove, making use of the orthogonality of P_x and N_x at any point $x\in\mathfrak{X}$ with respect to G, where $P_x=P(T_x(\mathfrak{X}))$ and N_x is the kernel of P on $T_x(\mathfrak{X})$.

Since

$$\begin{aligned}\frac{\partial\bar{g}_{il}}{\partial u^h}&=\bar{g}_{il;h}+\bar{g}_{kl}\{{}_i{}^k{}_h\}+\bar{g}_{ik}\{{}_l{}^k{}_h\}\\&=g_{qk}(P^q_{i;h}P^k_l+P^q_iP^k_{l;h})+\bar{g}_{kl}\{{}_i{}^k{}_h\}+\bar{g}_{ik}\{{}_l{}^k{}_h\},\end{aligned}$$

we have

$$\begin{aligned}\frac{\partial\bar{g}_{il}}{\partial u^h}Q^l_pg^{pj}&=[g_{qk}(P^q_{i;h}A^k_p+P^q_iA^k_{p;h}-P^q_iP^k_lQ^l_{p;h})+g_{ql}P^q_kA^l_p\{{}_i{}^k{}_h\}+\bar{g}_{ik}\{{}_l{}^k{}_h\}Q^l_p]g^{pj}\\&=A^j_kP^k_{i;h}-g_{qk}P^q_iN^k_{p;h}g^{pj}-\bar{g}_{il}Q^l_{p;h}g^{pj}+A^j_qP^q_k\{{}_i{}^k{}_h\}+\bar{g}_{ik}\{{}_l{}^k{}_h\}Q^l_pg^{pj}\\&=A^j_kP^k_{i;h}+g_{qk}P^q_{i;h}N^k_pg^{pj}-\bar{g}_{il}Q^l_{p;h}g^{pj}+P^j_k\{{}_i{}^k{}_h\}+\bar{g}_{ik}\{{}_l{}^k{}_h\}Q^l_pg^{pj};\end{aligned}$$

that is

$$\frac{\partial\bar{g}_{il}}{\partial u^h}Q^l_pg^{pj}=P^j_{i;h}+P^j_k\{{}_i{}^k{}_h\}+\bar{g}_{ik}(\{{}_l{}^k{}_h\}Q^l_p-Q^k_{p;h})g^{pj},$$

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3) Q is the homomorphism of $T(\mathfrak{X})$ such that it operates as $Q=P^{-1}$ on each $P(T_x(\mathfrak{X}))$, $x\in\mathfrak{X}$ and $Q=P$ on the kernel of P.

and

$$\frac{\partial \bar{g}_{hl}}{\partial u^i} Q^l_p g^{pj} = P^j_{h;i} + P^j_k \{{}_h{}^k{}_i\} + \bar{g}_{hk}(\{{}_l{}^k{}_i\} Q^l_p - Q^k_{p;i}) g^{pj}.$$

Since we have analogously

$$\frac{\partial \bar{g}_{ih}}{\partial u^l} Q^l_p g^{pj} = [g_{qk}(P^q_{i;l} P^k_h + P^q_i P^k_{h;l}) + \bar{g}_{kh}\{{}_i{}^k{}_l\} + \bar{g}_{ik}\{{}_h{}^k{}_l\}] Q^l_p g^{pj},$$

we get

$$\overline{[ih, l]} Q^l_p g^{pj} = \frac{1}{2}(P^j_{i;h} + P^j_{h;i}) + P^j_k \{{}_i{}^k{}_h\}$$
$$-\frac{1}{2}(\bar{g}_{ik} Q^k_{p;h} + \bar{g}_{hk} Q^k_{p;i}) g^{pj} - \frac{1}{2} g_{qk}(P^q_{i;l} P^k_h + P^q_i P^k_{h;l}) Q^l_p g^{pj}.$$

On the other hand, making use of the orthogonality of P_x and N_x, we have

$$-(\bar{S}_{lki} P^k_h + \bar{S}_{lkh} P^k_i) Q^l_p g^{pj} + \bar{S}_i{}^j{}_h$$
$$= -\frac{1}{2} g_{kq} A^q_t (P^t_{l;i} - P^t_{i;l}) P^k_h Q^l_p g^{pj} - \frac{1}{2} g_{kq}(P^q_{l;h} - P^q_{h;l}) P^k_i Q^l_p g^{pj}$$
$$+\frac{1}{2} A^j_l (P^l_{i;h} - P^l_{h;i}).$$

Substituting these equations into (1), we have

$$= \frac{1}{2}(P^j_{i;h} + P^j_{h;i}) + P^j_k \{{}_i{}^k{}_h\}$$
$$-\frac{1}{2}(\bar{g}_{ik} Q^k_{p;h} + g_{kq} P^q_{l;h} P^k_i Q^l_p) g^{pj} - \frac{1}{2}(\bar{g}_{hk} Q^k_{p;i} + g_{kq} P^q_{l;i} P^k_h Q^l_p) g^{pj}$$
$$+\frac{1}{2} A^j_l (P^l_{i;h} - P^l_{h;i}).$$

Since we have

$$(\bar{g}_{ik} Q^k_{p;h} + g_{kq} P^q_{l;h} P^k_i Q^l_p) g^{pj}$$
$$= g_{kq} P^k_i (P^q_l Q^l_{p;h} + P^q_{l;h} Q^l_p) g^{pj} = g_{kq} P^k_i A^q_{p;h} g^{pj} = -g_{kq} P^k_i N^q_{p;h} g^{pj}$$
$$= g_{kq} P^k_{i;h} N^q_p g^{pj} = N^j_k P^k_{i;h},$$

hence the above equation can be rewritten as

$$= \frac{1}{2} A^j_k (P^k_{i;h} + P^k_{h;i}) + P^j_k \{{}_i{}^k{}_h\} + \frac{1}{2} A^j_k (P^k_{i;h} - P^k_{h;i}),$$

that is

$$(3) \qquad \Gamma^j_{ih} = A^j_k P^k_{i;h} + P^j_k \{{}_i{}^k{}_h\}.$$

Accordingly we have

$$\Lambda^j_{ih} = \Gamma^j_{ih} - \frac{\partial P^j_i}{\partial u^h}$$
$$= A^j_k \left(\frac{\partial P^k_i}{\partial u^h} + \{{}_l{}^k{}_h\} P^l_i - \{{}_i{}^l{}_h\} P^k_l \right) + P^j_k \{{}_i{}^k{}_h\} - \frac{\partial P^j_i}{\partial u^h},$$

that is

$$(4) \qquad \Lambda^j_{ih} = \left(A^j_k \{{}_l{}^k{}_h\} - \frac{\partial A^j_l}{\partial u^h} \right) P^l_i.$$

Hence we obtain the equations

$$A^j_{ph}N^p_i=0,$$

which show $\Gamma N=0$. q.e.d.

Now, in [11] the author proved the following theorems.

Theorem C. *If a regular general connection*[4] $\Gamma=\partial u_j\otimes(P^j_i d^2u^i+\Gamma^j_{ih}du^i\otimes du^h)$ *is metric with respect to a non-singular symmetric tensor* $G=g_{ij}du^i\otimes du^j$ *and satisfies the condition*

$$(5) \qquad S_i{}^j{}_h=\frac{1}{2}(\Gamma^j_{ih}-\Gamma^j_{hi})=\frac{1}{2}(P^j_{i;h}-P^j_{h;i}),$$

then the covariant part ΓP^{-1} *of* Γ *is the Levi-Civita's connection made by* G.

Theorem D. *Let* Γ *be a metric regular general connection with respect to a non-singular symmetric tensor* $G=g_{ij}du^i\otimes du^j$ *on* $\mathfrak{X}$ *and* A *be a projection of* $T(\mathfrak{X})$ *such that* A_x *and* N_x *are invariant under* $P=\lambda(\Gamma)$ *and orthogonal with respect to* G *at each point* x *of* $\mathfrak{X}$, *where* $N=1-A$, $A_x=A(T_x(\mathfrak{X}))$ *and* $N_x=N(T_x(\mathfrak{X}))$. *If* Γ *satisfies the condition* (5) *in* **Theorem C**, *then* $\tilde{\Gamma}=A\Gamma A$ *is a normal, proper general connection which is metric with respect to* G *and* $\bar{G}=A(G)=g_{hk}A^h_iA^k_jdu^i\otimes du^j$ *and satisfies the generalized symmetric condition*:

$$\tilde{S}_k{}^j{}_hA^k_i=\frac{1}{2}A^j_l(\tilde{P}^l_{k;h}-\tilde{P}^l_{h;k})A^k_i,$$

where $\tilde{S}_i{}^j{}_h=\frac{1}{2}(\tilde{\Gamma}^j_{ih}-\tilde{\Gamma}^j_{hi})$ *and* $\tilde{P}=\partial u_j\otimes\tilde{P}^j_idu^i=\lambda(\tilde{\Gamma})$.

Another proof of **Theorem B.** $P^*=P+N$ is clearly regular and $(P^*)^{-1}=Q+N$. Let us denote the Levi-Civita's connection made by G by Γ_G. By means of Theorem A, there exists a uniquely determined regular metric general connection Γ^* such that $P^*=\lambda(\Gamma^*)$ and it satisfies the generalized symmetric condition with respect to G. By means of Theorem C, $\Gamma^*=\Gamma_GP+\Gamma_GN$. Furthermore, by means of Theorem D, the normal general connection

$$\Gamma=A\Gamma^*A=A(\Gamma_GP+\Gamma_GN)A$$
$$=A\Gamma_GPA+A\Gamma_GNA=A\Gamma_GP$$

is metric with respect to G and satisfies the generalized symmetric condition with respect to G. And we have

$$\lambda(\Gamma)=\lambda(A\Gamma_GP)=A1P=P$$

and $N\Gamma=NA\Gamma_GP=0$, $\Gamma N=A\Gamma_GPN=0$. Hence this connection Γ is the wanted one. Furthermore, we have

$$A\Gamma_G=(A^j_i,\ A^j_k\{{}_i{}^k{}_h\}),^{5)}$$

$$A\Gamma_GP=\left(P^j_i,\ A^j_k\{{}_l{}^k{}_h\}P^l_i+A^j_k\frac{\partial P^k_i}{\partial u^h}\right)$$

4) A general connection Γ is called regular when $P=\lambda(\Gamma)$ is an isomorphism of $T(\mathfrak{X})$.

5) See [11], §1.

and

$$\begin{aligned}\Gamma^{j}_{ih} &= A^{j}_{k}\left\{{k \atop l\ h}\right\}P^{l}_{i} + A^{j}_{k}\frac{\partial P^{k}_{i}}{\partial u^{h}} \\ &= A^{j}_{k}\left\{{k \atop l\ h}\right\}P^{l}_{i} + A^{j}_{k}\left(P^{k}_{i;h} - \left\{{k \atop l\ h}\right\}P^{l}_{i} + \left\{{l \atop i\ h}\right\}P^{k}_{l}\right) \\ &= A^{j}_{k}P^{k}_{i;h} + P^{j}_{k}\left\{{k \atop i\ h}\right\},\end{aligned}$$

which is identical with (3).

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