138. *On Evans Potential*

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1962)

Throughout this note, we always assume that R is an *open Riemann surface with null boundary* and $(R_n)_{n=0}^{\infty}$ is a normal exhaustion of R such that $R-\overline{R}_0$ is connected. An *Evans potential* $p(z)$ on R is a harmonic function with one negative logarithmic singularity at a point of R such that $\lim_{n\to\infty} \inf_{R-\overline{R}_n \ni z} p(z)=\infty$. Kuramochi[1] proved the existence of Evans potential on R. Although his argument is very interesting and contains wide generality, it is somewhat complicated and difficult to follow. So we give here a simple shorter alternating proof. We shall take Čech boundary as the ideal boundary of R, which makes automatically the Green kernel continuous[2] with respect to one variable. But we shall abandon to prove the symmetricity of Green kernel at the ideal boundary and to supply this disadvantage, we shall make some trick for the evaluation of the transfinite diameter of the ideal boundary. We shall prove the following

THEOREM. *There exists a harmonic function $u(z)$ on $R-\overline{R}_0$ with boundary value zero on ∂R_0 such that $\int_{\partial R_0} {}^*du=2\pi$ and*

$$\lim_{n\to\infty} \inf_{R-\overline{R}_n \ni z} u(z)=\infty.$$

In virtue of the linear operator method of Sario,[3] the existence of Evans potential follows at once from the above theorem. In fact, let L be a normal linear operator of Sario on $R-\partial R_0$ and $s(z)$ be the harmonic function with singularity such that $-s(z)$ is equal to the Green function $g_0(z,w)$ in R_0 with pole w in R_0 and $s(z)$ is equal to $u(z)$ in Theorem in $R-\overline{R}_0$. Since $\int_{\partial R_0^+ + \partial R_0^-} {}^*ds=0$, the equation $p-s=L(p-s)$ has a solution on R, which is an Evans potential on R.

1. Green kernel on Čech compactification. Let R^* be the *Čech compactification* of R, i.e. the compact Hausdorff space containing R as its dense subspace such that any bounded continuous function on R is continuously extended to R^*. Since R is completely regular, R^* exists uniquely.[4] Moreover, since R is locally compact,

1) Osaka Math. J., **8**, 119-137 (1956).

2) Hereafter, a continuous function means $[-\infty,\infty]$-valued continuous.

3) Trans. Amer. Math. Soc., **72**, 281-295 (1952).

4) E. Čech: Ann. of Math., **38**, 823-844 (1937).

R is open in R^*. We denote by Γ the compact set R^*-R, which is called the *Čech boundary* of R. We remark one more fact that any continuous function $f(z)$ on R is continuously extended to R^* uniquely. In fact, let $g(z)=\max(f(z), 0)$ and $h(z)=g(z)-f(z)$ on R. Then $(1+g(z))^{-1}$ and $(1+h(z))^{-1}$ are bounded continuous functions on R and so extended continuously to R^*. Hence the same is true for $g(z)$ and $h(z)$. We denote the extended functions by the same notations. Assume that $g(p)=\infty$ at a point p in R^*. Then we can find a neighborhood U of p such that $g>0$ on U and so $f>0$ on $U\cap R$ and $h=0$ on $U\cap R$. As R is dense in R^*, so $h(p)=0$. Similarly, $h(p)=\infty$ implies $g(p)=0$. Thus the expression $g(p)-f(p)$ on R^* has a definite meaning and gives a continuous extension to R^* of f. Since R is dense in R^*, the extension is unique.

Let $g(z, w)$ be the *Green function* on $R-\overline{R}_0$ with pole w. We set $g(z, w)=0$ if at least one of z and w belongs to $\overline{R}_0$. Then, since $g(z, w)=g(w, z)$ is continuous on $(R-R_0)\times(R-R_0)$, thus extended function $g(z, w)$ is continuous on $R\times R$. Hence if we fix one variable in R, $g(z, w)$ is continuously extended to R^* with respect to the other variable. For (z, p) in $R\times\Gamma$, we set

$$g(z, p)=\lim_{R\ni w\to p} g(z, w).$$

If we fix p in Γ, then $g(z, p)$ is continuous on R and harmonic on $R-\overline{R}_0$ and vanishes on $\overline{R}_0$. In fact, clearly $g(z, p)=0$ on $\overline{R}_0$. So we have to show that $g(z, p)$ is harmonic on $R-\overline{R}_0$ and vanishes continuously at ∂R_0. Let z_0 be a point in $R-\overline{R}_0$. Given an arbitrary positive number ε. By Harnack's inequality, we can find a disc K with center z_0 such that for any z in K and w in $R-\overline{R}_0-\overline{K}$, we have $|g(z, w)-g(z_0, w)|<\varepsilon$. Hence by letting $w\to p$, $|g(z, p)-g(z_0, p)|\leq\varepsilon$ for any z in K. This shows that $g(z, p)$ is continuous on $R-\overline{R}_0$. Next take a countable dense subset (z_k) in $R-\overline{R}_0$. Since $g(z_k, w)\to g(z_k, p)$ $(w\to p)$, we can find sequences $(U_{k,n})_{n=1}^{\infty}$ of neighborhoods of p such that $\bigcap_n(R\cap U_{k,n})=\phi$ and $U_{k,n}\supset U_{k,n+1}, U_{k+1,n}$ and $\lim_n \sup_{U_{k,n}\cap R\ni w}|g(z_k, w)-g(z_k, p)|=0$. Let $V_n=R\cap U_{n,n}$. Then $\lim_n \sup_{V_n\ni w}|g(z_k, w)-g(z_k, p)|=0$ for any $k=1, 2, 3, \cdots$. Fix a point w_n in V_n. The sequence $(g(z, w_n))_{n=1}^{\infty}$ converges to $g(z, p)$ on the dense set (z_k). Again by Harnack's inequality, it converges to a harmonic function $u(z)$ on $R-\overline{R}_0$. Hence $u(z)=g(z, p)$ on (z_k) and since $g(z, p)$ is continuous on $R-\overline{R}_0$, and $g(z, w_n)=0$ on ∂R_0,

$$g(z, p)=\lim_{n\to\infty} g(z, w_n) \tag{1}$$

uniformly on each compact subset of $R-R_0$.

Since $g(z, p)$ is continuous on R, it is extended continuously to R^*. From hitherto considerations, we may define *Green kernel* $G(p, q)$

on $(R^*-R_0)\times(R^*-R_0)$ by

$$G(p, q)=\lim_{R-R_0\ni z\to p}(\lim_{R-R_0\ni w\to q} g(z, w)).$$

From this definition, it is clear that $G(p, q)=g(p, q)$ in $R-R_0$, i.e. G is an extension of g. It is also clear that if at least one of p and q belongs to $R-R_0$, then G is symmetric, i.e.

$$(2)\qquad G(p, q)=G(q, p).$$

Notice that we do not claim $G(p, q)=G(q, p)$ for p and q in Γ. Since $G(z, q)=g(z, q)$ for (z, q) in $(R-R_0)\times\Gamma$, if we fix q in Γ, then $G(p, q)$ is continuous on R^*-R_0 and harmonic on $R-R_0$ and vanishes on ∂R_0. Again notice that we do not claim the continuity of $G(p, q)$ with respect to q at Γ for fixed p in Γ. Moreover we have for a fixed q in $R^*-\overline{R}_0$,

$$(3)\qquad \int_{\partial R_0} {}^*dG(z, q)=2\pi.$$

In fact, let $q\in R-\overline{R}_0$ and V be a disc with center q such that $\overline{V}\subset R_n-\overline{R}_0$ $(n>n_0)$ and $v_n(z)$ be harmonic in $F_n=R_n-\overline{R}_0-\overline{V}$ with boundary value 1 on $\partial R_0\cup\partial V$ and 0 on ∂R_n. By Green's formula, $D_{F_n}(v_n, G)=\int_{\partial R_0}{}^*dG+\int_{\partial V}{}^*dG$. Since $v_n\uparrow 1$ and $\int_{\partial V}{}^*dG=-2\pi$, we get $\int_{\partial R_0}{}^*dG(z, q)=2\pi$. Next suppose $q\in\Gamma$. By (1), we can find a sequence (w_m) in $R-\overline{R}_0$ such that $G(z, w_m)\to G(z, q)(m\to\infty)$ on $R-R_0$. Hence $\int_{\partial R_0}{}^*dG(z, q)=\lim_n \int_{\partial R_0}{}^*dG(z, w_n)=2\pi$.

2. Transfinite diameter of Γ. For each compact set K in R^*-R_0, we set

$$\binom{n}{2}D_n(K)=\inf_{p_1,\dots,p_n\in K}\sum_{i<j}^{1,\dots,n} G(p_i, p_j).$$

The sequence $(D_n(K))_{n=1}^{\infty}$ is non-decreasing, since the usual proof of this need not the symmetricity of kernel.[5] Hence we can define

$$D(K)=\lim_{n\to\infty} D_n(K).$$

Then $D(K)$ increases as K decreases. Similarly we set

$$n\,E_n(K)=\sup_{p_1,\dots,p_n\in K}\inf_{p\in K}\sum_{i=1}^{n} G(p, p_i).$$

It is clear that $(m+n)E_{m+n}(K)\geq mE_m(K)+nE_n(K)$. Hence the sequence $(E_n(K))_{n=1}^{\infty}$ converges and so we can define[5]

$$E(K)=\lim_{n\to\infty} E_n(K).$$

For these two quantities, we get

$$(4)\qquad E(K)\geq D(K).$$

In fact, fix a positive integer n and a point p_n in K. Since $G(p, p_n)$

5) See, for example, M. Tsuji: Potential Theory in Modern Function Theory, Maruzen, Tokyo (1959). See also M. Ohtsuka: Topics on Function Theory, Kyôritsu, Tokyo (1957) (in Japanese).

is continuous on R^*-R_0 in p, we can find a point p_{n-1} in K such that $G(p_{n-1}, p_n)=\inf_{p\in K}G(p, p_n)$ and the right term $\leq E_1(K)$ and so $G(p_{n-1}, p_n)\leq E_1(K)$. Similarly, since $G(p, p_{n-1})+G(p, p_n)$ is continuous on R^*-R_0 in p, we can find a point p_{n-2} in K such that $G(p_{n-2}, p_{n-1})+G(p_{n-2}, p_n)=\inf_{p\in K}(G(p, p_{n-1})+G(p, p_n))$ and the right hand term $\leq 2E_2(K)$ and so $G(p_{n-2}, p_{n-1})+G(p_{n-2}, p_n)\leq 2E_2(K)$. Repeating this process, we get n points $p_n, p_{n-1}, p_{n-2},\cdots, p_2, p_1$ in K such that

$$\sum_{j=n-i+1}^{n} G(p_{n-i}, p_j)\leq iE_i(K) \quad (i=1, 2,\cdots, n-1).$$

Summing up these $n-1$ inequalities, we get $\sum_{i<j} G(p_i, p_j)\leq\sum_{i=1}^{n-1} iE_i(K)$, or

$$D_n(K)\leq\left(\sum_{i=1}^{n-1} iE_i(K)\right)\Big/\binom{n}{2}.$$

By letting $n\to\infty$ in the above inequality, we get (4).

For simplicity, we set $K_m=R^*-R_m$ and $B_m=\partial R_m$. Then

$$(5) \qquad D(K_m)=D(B_m).$$

In fact, it is sufficient to show that $D_n(K_m)=D_n(B_m)$. Clearly $D_n(K_m)\leq D_n(B_m)$. So we have only to show $D_n(K_m)\geq D_n(B_m)$. For the aim, take n arbitrary points $p_{1,1}, p_{1,2},\cdots, p_{1,n}$ in K_m. The number $a_1=\sum_{i<j} G(p_{1,i}, p_{1,j})$ is the sum of $h_1(p_{1,1})=\sum_{j=2}^{n} G(p_{1,1}, p_{1,j})$ and $\sum_{i<j;\, i,\, j\neq 1} G(p_{1,i}, p_{1,j})$. The function $h_1(z)$ is positive harmonic on $R-R_m$ except at most a finite number of possible logarithmic pole. So $h_1(z)$ takes its minimum on $B_m=\partial R_m$. In fact, let $t=\min_{B_m} h_1(z)$. Contrary to the assertion, assume that $(z\in R-R_m;\, h_1(z)<s)\neq\phi$ for some $s<t$. Let F be a component of this set. Then, since $s-h_1(z)$ is a bounded harmonic function on F vanishing on ∂F and not constant, $F\notin SO_{HB}$. This is a contradiction, since $R\in O_G$.[6] Hence we can find a point $p_{2,1}$ in B_m such that $h_1(p_{2,1})\leq h_1(p_{1,1})$. Let $p_{2,j}=p_{1,j}(j\neq 1)$ and $a_2=\sum_{i<j} G(p_{2,i}, p_{2,j})$. Then $a_1\geq a_2$. Notice that $G(p_{2,1}, p_{2,2})=G(p_{2,2}, p_{2,1})$, since $p_{2,1}$ is in $R-R_0$. Then a_2 is the sum of $h_2(p_{2,2})=\sum_{j=1, j\neq 2}^{n} G(p_{2,2}, p_{2,j})$ and $\sum_{i<j;\, i,\, j\neq 2} G(p_{2,i}, p_{2,j})$. Similary as above, $h_2(z)$ takes its minimum on B_m and so we can find a point $p_{3,2}$ in B_m such that $h_2(p_{3,2})\leq h_2(p_{2,2})$. Let $p_{3,j}=p_{2,j}(j\neq 2)$ and $a_3=\sum_{i<j} G(p_{3,i}, p_{3,j})$. Then $a_2\geq a_3$. Repeating this process, we finally get n points $p_{n,1}, p_{n,2},\cdots, p_{n,n}$ in B_m such that

$$\sum_{i<j} G(p_{1,i}, p_{1,j})\geq\sum_{i<j} G(p_{n,i}, p_{n,j}),$$

which proves our assertion (5).

We define one more quantity only for B_m. Let (μ) be the family of unit Borel measures on B_m and $I_m(\mu)$ be the energy integral $\iint G(z, w)d\mu(z)d\mu(w)$ and set

$$W(B_m)=\inf_{\mu\in(\mu)} I_m(\mu).$$

Since $G(z, w)$ is a positive, symmetric and continuous kernel on B_m, it is well known that[5]

6) See, for example, T. Kuroda: Osaka Math. J., **6**, 231–241 (1954).

(6) $$D(B_m)=W(B_m).$$

Moreover, we have

(7) $$\lim_{m\to\infty}W(B_m)=\infty.$$

In fact, since $G(z,w)$ is a usual potential theoretic kernel on $(\overline{R}_m-R_0)\times(\overline{R}_m-R_0)$, there exists a μ_m in (μ) such that $I_m(\mu_m)=W(B_m)$ and the function $U_m(z)=\int G(z,w)d\mu_m(w)$ is equal to the constant $c_m=W(B_m)$ on B_m except a polar set in B_m.[5] Clearly $U_m(z)$ is harmonic on $R_m-\overline{R}_0$ and vanishes on ∂R_0 and so $U_m(z)=c_m w_m(z)$ in $R_m-\overline{R}_0$, where $w_m(z)$ is the harmonic function in $R_m-\overline{R}_0$ with $w_m=1$ on ∂R_m and $w_m=0$ on ∂R_0. By Green's formula and (3),

$$c_m D_{R_m-\overline{R}_0}(w_m)=c_m\int_{\partial R_m}{}^*dw_m=c_m\int_{\partial R_0}{}^*dw_m=\int_{\partial R_0}{}^*dU_m$$
$$=\int\Big(\int_{\partial R_0}{}^*d_zG(z,w)\Big)d\mu_m(w)=2\pi,$$

whence $c_m=2\pi/D_{R_m-\overline{R}_0}(w_m)\to\infty\,(m\to\infty)$, since $w_m\uparrow 1$.

3. **Proof of Theorem.** From (4), (5) and (6), $E(\Gamma)\geq D(\Gamma)\geq D(K_m)=D(B_m)=W(B_m)$ $(m=1,2,\cdots)$. Thus by (7), $E(\Gamma)=\infty$ or $\lim_{n\to\infty}E_n(\Gamma)=\infty$. Hence we can find a suitable subsequence $(n_k)_{k=1}^{\infty}$ such that $E_{n_k}(\Gamma)\geq 2^{k-1}(k=1,2,\cdots)$. Let μ_k be the Borel measure on Γ with total measure 2^{-k} such that $\mu_k(p_{k,i})=1/n_k2^k$ $(i=1,2,\cdots,n_k)$, where $p_{k,i}$ $(i=1,\cdots,n_k)$ are chosen in Γ so as to satisfy

$$\inf_{p\in\Gamma}\textstyle\sum_{i=1}^{n_k}G(p,p_{k,i})\geq n_k2^{k-1}.$$

This is possible by the choice of (n_k). Let

$$u_k(p)=\int G(p,q)d\mu_k(q)=\textstyle\sum_{i=1}^{n_k}G(p,p_{k,i})/n_k2^k.$$

Clearly $u_k(z)$ is harmonic in $R-\overline{R}_0$ and vanishes on ∂R_0 and continuous on R^*-R_0. So there exists a neighborhood V_k of Γ such that $u_k(p)>1/2$ on V_k, since $u_k(p)>1/2$ on Γ.

Here we remark that $G(z,q)=G(q,z)$ is finitely continuous for $(z,q)\in(R-R_0)\times\Gamma$. In fact, $G(z,q)$ is harmonic in $z\in(R-R_0)$ for fixed $q\in\Gamma$ and finitely continuous in $q\in\Gamma$ for fixed $z\in(R-R_0)$ and so by applying Harnack's inequality, we get our assertion.[7]

Let $\mu=\sum_{k=1}^{\infty}\mu_k$. Then μ is a Borel measure on Γ with $\mu(\Gamma)=1$. Set

$$u(z)=\int G(z,q)d\mu(q).$$

Then $u(z)$ is harmonic on $R-R_0$ and vanishes on ∂R_0. Clearly

$$u(p)=\textstyle\sum_{k=1}^{\infty}u_k(p)$$

on R^*-R_0. As R^*-R_n is a neighborhood of Γ and $\bigcap_{n=1}^{\infty}(R^*-R_n)=\Gamma$,

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7) Cf. Lemma 3.1 at p. 445 in M. Heins: Ann. of Math., **61** (1955).

so for each k, there exists n such that $R^*-R_n \subset V_1 \cap V_2 \cap \cdots \cap V_k$. Hence

$$\inf_{z \in R-R_m} u(z) > k/2 \quad (m \geq n).$$

Thus we have

$$\lim_n \inf_{R-R_n \ni z} u(z) = \infty.$$

By (3), we see that

$$\int_{\partial R_0} {}^*du = \int \left(\int_{\partial R_0} {}^*dG(z, q) \right) d\mu(q) = 2\pi.$$