10. A Converse Theorem on the Summability Methods

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§1. In a recent paper the author proved the following

Theorem 1. *If $\{s_n\}$ is summable (l) to s, then it is summable (L) to the same sum. There is a sequence summable (L) but not summable (l).* (See [5].)

Here we prove a converse of this theorem:

Theorem 2. *If $\{s_n\}$ is summable (L) to s, and if further $s_n \geq -M$, then it is summable (l) to the same sum.*

The latter theorem corresponds to the following celebrated theorem of Hardy and Littlewood:

Theorem 3. *If $\{s_n\}$ is Abel summable to s, and if further $s_n \geq -M$, then it is Cesàro summable $(C, 1)$ to the same sum.* (See [3], [2] Theorem 94.)

Here we use the same notations as before. When a sequence $\{s_n\}$ is given we define the method L as follows: If

$$\frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

tends to a finite limit s as $x \to 1$ in the open interval $(0, 1)$, we say that $\{s_n\}$ is summable (L) to s and write $\lim s_n = s(L)$. (See [1].)

On the other hand we define the method l as follows: If

$$t_0 = s_0,\ t_1 = s_1,$$

$$t_n = \frac{1}{\log n}\left(s_0 + \frac{s_1}{2} + \cdots + \frac{s_n}{n+1}\right) \quad (n \geq 2)$$

tend to a finite limit s as $n \to \infty$, we say that $\{s_n\}$ is summable (l) to s and write $\lim s_n = s(l)$. (See [2] p. 59, p. 87.)

§2. **Proof of Theorem 2.** For the proof we use the method of Karamata [6]. (See also [2] pp. 156–158, [7] pp. 55–57.) Without loss of generality we may assume that the s_n are non-negative, for otherwise we would work with the sequence $s_n + M$ which is non-negative. At first we shall prove two lemmas.

Lemma 1. Let $g(x)$ be continuous except at most for one discontinuity of the first kind in the closed interval $[0, 1]$. Let further $g(x)$ be bounded in $[0, 1]$, and $g(x) = g(x+0)$ and $g(1) = g(1-0)$. Then to every positive ε, there exist two polynomials, $p(x)$ and $q(x)$, such that

(1) $$q(x) \leq g(x) \leq p(x) \quad for \quad 0 \leq x \leq 1,$$

and

(2) $$0 \leq g(x) - q(x) \leq \varepsilon, \quad 0 \leq p(x) - g(x) \leq \varepsilon$$
for $0 < \sigma \leq x \leq 1$, *where* σ *is an appropriate positive constant,* $0 < \sigma < 1$.

Proof of Lemma 1. At first let $g(x)$ be continuous. Then $g(x) \pm \frac{\varepsilon}{2}$ is continuous and by Weierstrass' Approximation Theorem (see [4] p. 228, [7] p. 55) there exist two polynomials $p(x)$, $q(x)$ such that

$$\left| q(x) - \left(g(x) - \frac{\varepsilon}{2} \right) \right| \leq \frac{\varepsilon}{2},$$

$$\left| p(x) - \left(g(x) + \frac{\varepsilon}{2} \right) \right| \leq \frac{\varepsilon}{2} \quad \text{for} \quad 0 \leq x \leq 1.$$

These two polynomials satisfy (1) and (2).

Next if $g(x)$ has a finite jump at $x = \xi$, $0 < \xi < 1$, we construct two functions $g^*(x)$ and $g_*(x)$ continuous in the closed interval $[0, 1]$ such that

$$g_*(x) \leq g(x) \leq g^*(x) \quad \text{for} \quad 0 \leq x \leq 1,$$

and

$$g_*(x) = g(x) = g^*(x) \quad \text{for} \quad 0 < \sigma \leq x \leq 1,$$

where we may take $\sigma = \frac{1}{2}(1+\xi)$ for example. Then to every positive ε, there exist two polynomials, $p(x)$ and $q(x)$, such that

$$q(x) \leq g_*(x), \quad g^*(x) \leq p(x)$$
$$0 \leq g_*(x) - q(x) \leq \varepsilon, \quad 0 \leq p(x) - g^*(x) \leq \varepsilon$$

for $0 \leq x \leq 1$. These two polynomials satisfy (1) and (2), whence the proof is complete.

Lemma 2. Let $g(x)$ *be any function of the type prescribed in Lemma 1. Then*

(3) $$\lim_{x \to 1-0} \frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} g(x^{n+1}) = s g(1),$$

where

$$s = \lim_{x \to 1-0} \frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}.$$

Proof of Lemma 2. At first we shall prove (3) when $g(x)$ is a non-negative power of x, i.e. $g(x) = x^c$ $(c \geq 0)$. In this case the left member of (3) is

$$\lim_{x \to 1-0} \frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} x^{c(n+1)}$$

$$= \lim \frac{-1}{\log(1-x)} \sum \frac{s_n}{n+1} x^{(c+1)(n+1)}$$

$$= \lim \frac{\log(1-x^{c+1})}{\log(1-x)} \cdot \frac{-1}{\log(1-x^{c+1})} \sum \frac{s_n}{n+1} x^{(c+1)(n+1)}$$

$$= \lim_{x \to 1-0} \frac{\log(1-x^{c+1})}{\log(1-x)} \cdot s = s = s \cdot 1^c.$$

Hence (3) is true whenever $g(x)$ is a polynomial. To prove the general case we use Lemma 1. Since we have assumed s_n non-negative, we have

$$\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}g(x^{n+1})\le\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}p(x^{n+1})$$

and

$$\varlimsup_{x\to1-0}\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}g(x^{n+1})\le\lim_{x\to1-0}\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}p(x^{n+1})$$

$$=sp(1)\le s\{g(1)+\varepsilon\}.$$

Inasmuch as ε may be taken arbitrary small, we have

$$\varlimsup_{x\to1-0}\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}g(x^{n+1})\le sg(1).$$

Similarly we get

$$\varliminf_{x\to1-0}\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}g(x^{n+1})\ge sg(1),$$

whence the proof is complete.

We shall now put

$$g(x)=\begin{cases}0 & \text{for}\quad 0\le x<\dfrac{1}{e}\\[2mm] \dfrac{1}{x} & \text{for}\quad \dfrac{1}{e}\le x\le 1.\end{cases}$$

Then we get $g(1)=1$, and further

$$g(x^{n+1})=0\quad\text{if}\quad x^{n+1}<\frac{1}{e},$$

i.e. if $n+1>\dfrac{1}{\log\frac{1}{x}}$.

Thus from (3)

$$\lim_{x\to1-0}\frac{-1}{\log(1-x)}\sum_{n\le\frac{1}{\log\frac{1}{x}}-1}\frac{s_n}{n+1}x^{n+1}\cdot\frac{1}{x^{n+1}}$$

$$=\lim_{x\to1-0}\frac{-1}{\log(1-x)}\sum_{n\le\frac{1}{\log\frac{1}{x}}-1}\frac{s_n}{n+1}=s.$$

If we put $x=e^{-\frac{1}{N}}$, we have

$$\lim_{N\to\infty}\frac{-1}{\log(1-e^{-\frac{1}{N}})}\sum_{n=0}^{N-1}\frac{s_n}{n+1}=s.$$

Since

$$\lim_{N\to\infty}\frac{-\log(1-e^{-\frac{1}{N}})}{\log N}=1,$$

we get

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=0}^{N-1}\frac{s_n}{n+1}=s.$$

Since $\lim_{N\to\infty}\dfrac{\log N}{\log (N-1)}=1$, we have $\lim_{n\to\infty} s_n = s(l)$.

This completes the proof of Theorem 2.

References

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