8. A Subset Homeomorphic to the Whole Space

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Professor Nagami suggests the problem:[1] Is there a nowhere dense subset which is homeomorphic to the underlying space of an infinite 0-dimensional compact T_1-group?

The purpose of this paper is to show that this problem is affirmative.

In this paper the dimension means the covering dimension which will be denoted by dim.

Lemma (L. N. Ivanovskii [3], V. Kuz'minov [4], A. Hulanicki [2], E. Hewitt [1]). *An infinite compact* T_1*-group* G *with* $\dim G=0$ *is homeomorphic to the generalized Cantor set* $\{0,1\}^\tau$, *where* τ *is the topological weight of* G.

Theorem. *An infinite compact* T_1*-group* G *with* $\dim G=0$ *has a nowhere dense subset, which is homeomorphic to* G.

Proof. Take a basis whose cardinal is τ, consisting of open and closed sets in G. Well order it as follows, $\{U_1, U_2,\cdots, U_\alpha,\cdots;\alpha<\vartheta\}$, where ϑ is the first ordinal with the cardinal τ. For each $\alpha<\vartheta$, φ_α denotes the characteristic function of U_α. We define a mapping Φ of G into $\{0,1\}^\tau$ by: $\Phi(x)=(\varphi_\alpha(x))_{\alpha<\vartheta}$. Since all the sets U_α are open and closed, Φ is a continuous mapping. Since the sets U_α separate points of G, Φ is one-to-one. Therefore Φ is a homeomorphism, and $\Phi(G)$ is a compact and hence closed subspace of $\{0,1\}^\tau$. It is impossible that $\Phi(x)=0$ for all but a finite number of α's at any point x in G. This implies that $\Phi(G)$ has no open subset in $\{0,1\}^\tau$.

By the lemma, there exists a homeomorphism Ψ of $\{0,1\}^\tau$ onto G, and $\Psi(\Phi(G))$ is nowhere dense in G.

Corollary 1. *Let* G *be a locally compact* T_1*-group with* $\dim G=0$ *which has no isolated point. Then* G *has a nowhere dense subset which is homeomorphic to* G.

This follows immediately from the well-known fact that G has an infinite compact open subgroup.

Corollary 2. *If a locally compact* T_1*-group* G *with* $\dim G<\infty$ *is not locally connected, then the invariance theorem of domain does not hold in* G.

Since we have the theorem, this is proved by the quite analo-

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1) In case of an infinite 0-dimensional compact metric T_1-group the answer of this problem is affirmative, which was proved in the proof of K. Nagami [5, Theorem 4.2].

gous method to the proof which was used in Nagami [5, Theorem 4.2].

References

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