4. On the Existence and the Propagation of Regularity of the Solutions for Partial Differential Equations. II

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3. **Main theorems.** Let us resolve L_0 in (1.3) into

$$L_0(t,x,\lambda,\sqrt{-1}\,\eta|\eta|^{-1})=\prod_{i=1}^{k}(\lambda+\lambda_{0,i}^{(1)}(t,x,\eta))\prod_{j=1}^{m-k}(\lambda+\lambda_{0,j}^{(2)}(t,x,\eta)) \tag{3.1}$$
$$(m\geqq k>0,\text{[6]}\ \eta\neq 0)$$

such that $|\Re e^{7)}\lambda_{0,j}(t,x,\eta)|\geqq\delta>0$ $(j=1,\cdots,m-k)$ with a constant δ. Then, we can write

$$L_0(t,x,\lambda,\sqrt{-1}\,\xi)=\prod_{i=1}^{k}(\lambda+\lambda_{0,i}^{(1)}(t,x,\xi R^{-1})r^{1/m})\prod_{j=1}^{m-k}(\lambda+\lambda_{0,j}^{(2)}(t,x,\xi R^{-1})r^{1/m})$$

with $r=r(\xi)$ defined by (2.1) and R defined by (2.4); see [4].

Theorem 1. *Let L be a differential operator of the form* (1.1) *with bounded measurable coefficients in a neighborhood of the origin, and assume that the coefficients of L_0 are in C^∞.*

Suppose that $\lambda_{0,i}^{(1)}(t,x,\eta)$ $(i=1,\cdots,k)$ are in $C^\infty_{(t,x,\eta)}(\eta\neq 0)$ and distinct, and each $\lambda_i(t,x,\xi)=\lambda_{0,i}^{(1)}(t,x,\xi R^{-1})r^{1/m}$ satisfies the condition

$$\frac{\partial}{\partial t}p_i+\sum_{j=1}^{\nu}\left\{\frac{\partial}{\partial x_j}p_i\frac{\partial}{\partial x_j}q_i-\frac{\partial}{\partial x_j}q_i\frac{\partial}{\partial \xi_j}p_i\right\}=\sigma(H_i)p_i\quad(|\xi|\geqq 1) \tag{3.2}$$

for $p_i=\Re e\,\lambda_i$, $q_i=\Im m\,\lambda_i$ and some $H_i(t)\in C^m_{\rm m}$. Then, with $\varphi_0=(1+t/2h_0)$ we have a priori inequality

$$\begin{aligned}&n\sum_{i+j=m-1}\int\varphi_0^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\right\|^2dt\\ &\leqq C\left\{\int\varphi_0^{-2n}\|Lu\|^2dt+\sum_{i+j=\tau\leqq m-2}n^{2(m-\tau)-1}\int\varphi_0^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\right\|^2dt\right\}\end{aligned} \tag{3.3}$$
$$u\in C_0^\infty(\Omega_{h_0})\text{[8]}$$

for a sufficiently small fixed h_0 and every $n(\geqq 1)$.

Remark. i) If $P_i\equiv 0$ or $P_i\neq 0$ for any $\xi\neq 0$, the condition (3.2) is always satisfied. ii) Here we do not require the regularity of $\lambda_{0,j}^{(2)}$ $(j=1,\cdots,m-k)$, but in the case when $\lambda_{0,j}^{(2)}$ are in $C^\infty_{(t,x,\eta)}$ $(\eta\neq 0)$ and distinct the uniqueness of the Cauchy problem holds; see [4].

Proof of Theorem 1. Let us write $\prod_{j=1}^{m-k}(\lambda+\lambda_{0,j}^{(2)}(t,x,\eta))=\sum_{j=0}^{m-k}h_{0,j}(t,x,\eta)\lambda^{m-k-j}$ $(h_{0,0}=1)$. Then, from the infinite differentiability of the

6) In the case when we can take $k=0$, L is *hypoelliptic* if the coefficients are in C^∞, and the existence theorem of solutions is easy from Lemma 3 for sufficiently small h. Hence, we may consider only the case $k>0$.

7) For a complex number a, by $\Re e\,a$ we shall denote the real part of a and by $\Im m\,a$ the imaginary part.

8) $\Omega_h=\{(t,x);\ t^2+K(x)^2<h^2\}$.

coefficients of L_0 and of $\lambda_{0,i}^{(1)}$ $(i=1,\cdots,k)$ it follows that $h_{0,j}(t,x,\eta)$ are in $C^\infty_{(t,x,\eta)}$ $(\eta\neq 0)$. So we have

$$L_0(t,x,\lambda,\sqrt{-1}\,\xi)=\prod_{i=1}^{k}(\lambda+\lambda_{0,i}^{(1)}(t,x,\xi R^{-1})r^{1/m})\Big(\sum_{j=0}^{m-k}h_{0,j}(t,x,\xi R^{-1})r^{j/m}\lambda^{m-k-j}\Big)$$

and with a positive constant δ

$$\Big|\sum_{j=0}^{m-k}h_{0,j}(t,x,\xi R^{-1})r^{j/m}(\sqrt{-1}\,\lambda)^{m-k-j}\Big|^2\geqq\delta^2(\lambda^{2(m-k)}+K(\xi)^{2(m-k)}). \tag{3.4}$$

For $u\in C_0^\infty(\Omega_h)$ (h; sufficiently small) we may consider operators $H_i^{(1)}$ $(i=1,\cdots,k)$ and $H_j^{(2)}$ $(j=1,\cdots,m-k)$ of class $C_{\mathfrak{m}}^m$ with $\sigma(H_i^{(1)})=\lambda_{0,i}(t,x,\xi R^{-1})$ and $\sigma(H_j^{(2)})=h_{0,j}(t,x,\xi R^{-1})$ respectively; see [3] p. 206.

Set $A_1=J_1\cdots J_k$ for $J_i=\partial/\partial t+H_i^{(1)}\Lambda$ $(i=1,\cdots,k)$ and $A_2=\sum_{j=0}^{m-k}H_j^{(2)}\Lambda^j\,\partial^{m-k-j}/\partial t^{m-k-j}$ $(H_0^{(2)}=1)$. Then, by the assumption (3.2) we can apply Lemma 1 to A_1 and get

$$n\sum_{i+j=k-1}\int\varphi_0^{-2n}\Big\|\frac{\partial^i}{\partial t^i}\Lambda^jA_2u\Big\|^2dt\leqq C\int\varphi_0^{-2n}\|A_1(A_2u)\|^2\,dt \qquad u\in C_0^\infty(\Omega_{h_0}) \tag{3.5}$$

for sufficiently small fixed h_0 and every $n(\geqq 1)$. On the other hand by the assumption (3.4) we can apply Lemma 3 to A_2 with the form $A_2(\partial^i/\partial t^i\,\Lambda^jv)$ $(i+j=k-1,\ v=\varphi_0^{-n}u)$, and get

$$\sum_{i+j=m-1}\Big|\Big|\Big|\frac{\partial^i}{\partial t^i}\Lambda^jv\Big|\Big|\Big|^2\leqq C\Big(\sum_{i+j=k-1}\Big|\Big|\Big|A_2\frac{\partial^i}{\partial t^i}\Lambda^jv\Big|\Big|\Big|^2+\sum_{i+j\leqq m-2}\Big|\Big|\Big|\frac{\partial^i}{\partial t^i}\Lambda^jv\Big|\Big|\Big|^2\Big). \tag{3.6}$$

From easy application of the Fourier transform we get

$$\begin{aligned}&\Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u\Big\|^2=\|\xi^\alpha\hat{u}(\xi)\|^2\leqq\|K(\xi)^{m|\alpha:\mathfrak{m}|}\hat{u}(\xi)\|^2=\|\Lambda_0^{m|\alpha:\mathfrak{m}|}u\|^2,\\ &h^{-2(a-b)}\|\Lambda_0^bu\|^2\leqq C_a\|\Lambda_0^au\|^2\quad(0\leqq b\leqq a,\ u\in C_0^\infty(K(x)<h)).\end{aligned} \tag{3.7}$$

Hence, by the theorem for the commutators of singular integral operators (see [3] p. 184), we have

$$\sum_{i+j=k-1}\Big\|\Big(\frac{\partial^i}{\partial t^i}\Lambda^jA_2-A_2\frac{\partial^i}{\partial t^i}\Lambda^j\Big)u\Big\|^2\leqq C\sum_{i+j\leqq m-2}\Big\|\frac{\partial^i}{\partial t^i}\Lambda^ju\Big\|^2$$

and

$$\|(L-A_1A_2)u\|^2=\|(L-L_0)u+(L_0-A_1A_2)u\|^2\leqq C\sum_{i+j\leqq m-1}\Big\|\frac{\partial^i}{\partial t^i}\Lambda^ju\Big\|^2.$$

Replacing v by $\varphi_0^{-n}u$ in (3.6) and using (3.5) and the above inequality we get (3.3). Q.E.D.

Now using (3.7) we have for $u\in C_0^\infty(\Omega_h)$

$$\sum_{i+m|\alpha:\mathfrak{m}|=\tau\leqq m-1}h^{-2(m-1-\tau)}\Big|\Big|\Big|\frac{\partial^{i+|\alpha|}}{\partial t^i\partial x^\alpha}u\Big|\Big|\Big|^2\leqq C\sum_{i+j=m-1}\Big|\Big|\Big|\frac{\partial^i}{\partial t^i}\Lambda^ju\Big|\Big|\Big|^2,$$

so that if we take sufficiently small h $(\leqq h_0)$ depending on fixed n such as $1/2\leqq\varphi_n^{-2n}\leqq 2$ for every t $(-h<t<h)$, then we have by (3.3)

$$n\sum_{i+m|\alpha:\mathfrak{m}|=\tau\leqq m-1}h^{-2(m-\tau-1)}\Big|\Big|\Big|\frac{\partial^{i+|\alpha|}}{\partial t^i\partial x^\alpha}u\Big|\Big|\Big|^2\leqq C|||Lu|||^2\quad(u\in C_0^\infty(\Omega_h)).$$

This shows that L^{-1} is bounded, so that there exists at least one weak solution of L^{*}[9] $u=f$ for $f\in L^2(\Omega_h)$.

Theorem 2. *Let L have the form* (1.1) *with the coefficients in* C^∞ *and the inequality* (3.3) *hold for this L. Suppose* $Lu=f$ *for* $f\in C^\infty$, *and u belongs to* C^∞ *in the compliment of a strictly convex set,*[10] *then u is in* C^∞ *in a neighborhood of the origin.*

Here we do not prove this, but we remark that if we transform t by $\theta=\log(1+t/2h_0)$, we get by (3.3)

$$n\sum_{i+j=m-1}\int e^{-2n\theta}\left\|\frac{\partial^i}{\partial\theta^i}\Lambda_0^j u\right\|^2 d\theta$$

$$\leqq C\left\{\int e^{-2n\theta}\|Lu\|^2 d\theta+\sum_{i+j=\tau\leqq m-2} n^{2(m-\tau)-1}\int e^{-2n\theta}\left\|\frac{\partial^i}{\partial\theta^i}\Lambda_0^j u\right\|^2 d\theta\right\}$$

$$u\in C_0^\infty(\Omega'_h)$$

where $\Omega'_h=\{(\theta,x);\,\theta^2h_0^2+K(x)^2<h_0^2\}$ (c.f. [2]).

References

[1] A. P. Calderón and A. Zygmund: Singular integral operators and differential equations, Amer. J. Math., **80**, 901-921 (1957).

[2] L. Hörmander: Differential operators with nonsingular characteristics, Bull. Amer. Math. Soc., **67**, 354-359 (1961).

[3] H. Kumano-go: On the uniqueness of the solution on the Cauchy problem and the unique continuation theorem for elliptic equation, Osaka Math. J., **14**, 181-212 (1962).

[4] H. Kumano-go: On the uniqueness for the solution of the Cauchy problem, Osaka Math. J., to appear.

[5] M. Matsumura: Existence des solution locales pour quelques opérateurs différentiels, Proc. Japan Acad., **37**, 383-387 (1961).

[6] S. Mizohata: Systèmes hyperboliques, J. Math. Soc. Japan, **11**, 205-233 (1959).

9) L^* means the formal adjoint operator of L.

10) By "*strictly convex set*" we mean a set which lies in $\{(t,x);\,t>0\}$ and of which *closure* meets the plane $(t=0)$ only at the origin.