2. *Existence of Pseudo-Analytic Differentials on Riemann Surfaces. II*

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III. Existence theorems. 1. Definition 3.1. *Let γ be an analytic closed curve on R, and ω be a real differential of C. The integral*

$$\int_{\gamma} \frac{1}{\sqrt{\sigma}} \omega \tag{3.1}$$

is called the σ-period of ω over γ, and denoted by $P_\sigma(\omega;\gamma)$. A differential ω is called σ-exact if all its σ-periods vanish.

The σ-exact differential can be written as Du for $u \in C^1$.

Theorem 3.1. *Let γ be an analytic closed curve which does not devide R, then there exists a differential $\omega \in H$ such that $P_\sigma(\omega;\gamma)=1$ and σ-exact in $R-\gamma$.*

Proof. We can construct a closed differential $\eta \in C^2 \cap L^2$ which is σ-exact in $R-\gamma$ and $P_\sigma(\eta:\gamma)=1$. Set $\eta_1=\sqrt{\sigma}\eta$, then $D_1\eta_1=0$. Therefore we have $\eta_1=\omega_h+\omega_1$ with $\omega_h \in H$, and $\omega_1 \in E$. Since $\eta_1 \in C_1$, we have $\omega_1 \in C^1$ and therefore $\omega_1=Du$ with $u \in C^2$. In $R-\gamma$, we have $\omega=\eta_1-Du$, and hence ω is σ-exact there. Moreover we have

$$\int_{\gamma} \frac{1}{\sqrt{\sigma}}\omega = \int_{\gamma} \frac{1}{\sqrt{\sigma}}(\eta_1 - Du) = \int_{\gamma}(\eta - du) = \int_{\gamma}\eta = 1.$$

2. Let $F(p)$ and $G(p)$ be the functions of $C^{1+\alpha}$ satisfying

$$-i\bar{F}G > 0 \quad \text{and} \quad M \geqq |F|+|G| \geqq M^{-1} > 0. \tag{3.2}$$

An (F,G)-pseudo-analytic function is an $[a,b]$-analytic function with

$$a = -\frac{\bar{F}G_{\bar{z}} - F_{\bar{z}}\bar{G}}{F\bar{G}-\bar{F}G}, \quad b = \frac{FG_{\bar{z}} - F_{\bar{z}}G}{F\bar{G}-\bar{F}G}, \tag{3.3}$$

and an (F,G)-pseudo-analytic differential is an $[a,b]$-analytic differential with

$$a = -\frac{\bar{F}G_{\bar{z}} - F_{\bar{z}}\bar{G}}{F\bar{G}-\bar{F}G}, \quad b = -\frac{FG_{z} - F_{z}G}{F\bar{G}-\bar{F}G}. \tag{3.4}$$

Under the condition (3.2), (F,G)-analytic function of the 2nd kind $\chi(p)=u(p)+iv(p)$ satisfy the equation

$$\begin{cases} v_x = -\sigma u_y \\ v_y = \sigma u_x \end{cases} \qquad \sigma = i\frac{F}{G} > 0. \tag{3.5}$$

Since $\sigma \in C^{1+\alpha}$, $u(p)$ is in C^2, and hence u is σ-harmonic.

3. We fix the point $p_0 \in R$, a neighborhood V of p_0, and its local parameter z. Let $W_0(z)$ be the (F,G)-analytic function similar to the function $1/z^n$ $(n \geqq 1)$ in V. Let $\chi_0(z)=u_0+iv_0$ be the (F,G)-analytic

function of the 2nd kind corresponding to $W_0(z)$. Then $\omega_0=Du_0=\sqrt{\sigma}\,du_0$ is a σ-harmonic differential in V. We shall prove

Theorem 3.2. *There exists a unique differential ω having the following properties*:

(1) *ω is σ-harmonic in $R-p_0$, and σ-exact in $R-p_0$.*

(2) *In a neighborhood V_1 of p_0, $\omega-\sqrt{\sigma}\,du_0$ is σ-harmonic.*

(3) *Let $h(p)\in C^2\cap L^2$ such that $h(p)\equiv 0$ in V_1, then $(\omega, Dh)=0$.*

(4) $\|\omega\|_{R-V_1}<\infty$.

Proof. Let V_1 and V_2 be the neighborhoods such that $V_1\Subset V_2\Subset V$. We define the real function $\rho(z)\in C^3(V)$ such that $\rho(z)\equiv 1$ in V_1, and $\rho(z)\equiv 0$ in $V-V_2$. We set

$$\eta_1=\begin{cases} D(\rho u_0) & \text{in } V \\ 0 & \text{in } R-V, \end{cases} \qquad \eta_2=\begin{cases} \frac{1}{\sigma}*D(\rho v_0) & \text{in } V \\ 0 & \text{in } R-V, \end{cases} \tag{3.6}$$

and

$$\eta=\eta_1+\eta_2. \tag{3.7}$$

We have $\eta=\sqrt{\sigma}\,du_0+\frac{1}{\sqrt{\sigma}}*dv_0=\sqrt{\sigma}\,du_0-\sqrt{\sigma}\,du_0=0$ in V_1, and $\eta\equiv 0$ in $R-V_2$. Hence $\eta\in L^2\cap C^{1+\alpha}$. From Theorem 2.3, we have $\eta=\omega_h+Du+\frac{1}{\sigma}*Dv$ with $\omega_h\in H$, $u, v\in C^2$, $Du\in\widetilde{E}$ and $\frac{1}{\sigma}*Dv\in E^*$. We set

$$\omega=\eta_1-Du=-\eta_2+\omega_h+\frac{1}{\sigma}*Dv. \tag{3.8}$$

We shall prove that ω satisfies the conditions (1)—(4). Proof of (1): $\omega\in C^1(R-p_0)$ and we have $D_1\omega=D_1\eta_1-D_1Du=d\Big(\frac{1}{\sqrt{\sigma}}\sqrt{\sigma}\,d(\rho u_0)\Big)-d\Big(\frac{1}{\sqrt{\sigma}}\sqrt{\sigma}\,du\Big)=0$ and $D_2*\omega=-D_2*\eta_2+D_2*\omega_h-D_2\Big(\frac{1}{\sigma}Dv\Big)=d\Big(\frac{1}{\sqrt{\sigma}}\sqrt{\sigma}\,d(\rho v_0)\Big)+D_2*\omega_h-D_2\Big(\frac{1}{\sigma}Dv\Big)=0$. Hence ω is σ-harmonic in $R-p_0$. Since $\omega=\eta-Du$, ω is σ-exact in $R-p_0$. Proof of (2): Since $\eta_1=-\eta_2=\sqrt{\sigma}\,du_0$ in V_1, we have $D_1(\omega-\sqrt{\sigma}\,du_0)=-D_1Du=0$ and $D_2*(\omega-\sqrt{\sigma}\,du_0)=D_2*\omega_h-D_2\Big(\frac{1}{\sigma}Dv\Big)=0$, hence $\omega-\sqrt{\sigma}\,du_0$ is σ-harmonic in V_1. Proof of (3): Since $h\in\widetilde{E}$, and $\omega_h\in\widetilde{H}$, we have $(\omega, Dh)=(-\eta_2, Dh)+(\omega_h, D_h)+\Big(\frac{1}{\sigma}*Dv, Dh\Big)=0$. Proof of (4): Since $\|\eta_1\|_{R-V_1}<\infty$ and $\|Du\|_R<\infty$, we have $\|\omega\|_{R-V_1}=\|\eta_1-Du\|_{R-V_1}\leqq\|\eta_1\|_{R-V_1}+\|Du\|_R<\infty$.

Finally, we shall prove the uniqueness. Let ω' be another differential which has the properties (1)—(4). Then $\omega'-\omega$ is in L^2, it is σ-harmonic on R and σ-exact. There is a function $u(p)\in C^2$ such that $\omega'-\omega=Du$. Consider the function $h(p)$ such that $h=\rho u$ in V and $h\equiv 0$ in $R-V$. Then $h-u$ has the properties of (3). Hence we have $(\omega'-\omega, D(h-u))=0$. Since $\omega'-\omega\in\widetilde{H}$ and $Dh\in\widetilde{E}$, we have $\|\omega'-\omega\|^2=$

$(\omega'-\omega, Du)=(\omega'-\omega, D(u-h))+(\omega'-\omega, Dh)=0.$

Theorem 3.3. *There exists a unique differential* $\varphi=fdz$ *such that*

(1) φ *is* (F, G)*-analytic in* $R-p_0$, *and corresponding* (F, G)*-analytic function of the* 2nd *kind has the single-valued real part in* $R-p_0$.

(2) φ *is similar to the analytic differential* $d(1/z^n)$ $(n\geqq 1)$ *in* V.

(3) $\|\varphi\|_{R-V}<\infty$.

Proof. Let ω be the differential obtained in the previous theorem. Since ω is σ-exact in $R-p_0$: $\omega=\sqrt{\sigma}\,du$ with σ-harmonic u, we can find a function $v(z)$ such that $dv=\sigma * du$ in every neighborhood V of $R-p_0$. Hence, $d\chi=du+i\sigma * du$ is the differential of an (F, G)-analytic function of the 2nd kind $\chi(z)$ in V. Therefore, the differential $\varphi=F\frac{1}{\sqrt{\sigma}}\omega+G\sqrt{\sigma} * \omega=Fdu+Gdv$ is an (F, G)-analytic differential of $R-p_0$. From the way of construction of ω, we know that φ has a singularity similar to $d(1/z^n)$ at p_0. Finally, we have $\|\varphi\|_{R-V_1}<\infty$.

4. In the next place, we fix the points p_0, q_0 in a neighborhood V. Let $w_0=Fu_0+Gv_0$ be the (F, G)-analytic function similar to the function $\log(z-A)/(z-B)$ in V, A and B being the values of z corresponding to p_0 and q_0 respectively. The slight modification of the proof of Theorem 3.2 and Theorem 3.3 gives the following

Theorem 3.4. *There exists a unique differential* ω *which has the following properties*:

(1) $\omega-\sqrt{\sigma}\,du_0$ *is* σ*-harmonic in* V, *and* ω *has the same* σ*-periods as* $\sqrt{\sigma}\,du_0$ *in* V.

(2) ω *is* σ*-harmonic in* $R-(p_0\cup q_0)$ *and is* σ*-exact in* $R-V$.

(3) $\|\omega\|_{R-V}<\infty$.

(4) *If* $h\in C^2\cap L^2$ *and* $h\equiv 0$ *in* V, *then* $(\omega, Dh)=0$.

Theorem 3.5. *There exists a differential* $\varphi=fdz$ *such that*

(1) φ *is* (F, G)*-analytic in* $R-(p_0\cup q_0)$, *and corresponding* (F, G)*-analytic function of the second kind* $\chi(p)$ *has the single-valued real part in* $R-V$.

(2) $\varphi-(Fdu_0+Gdv_0)$ *is* (F, G)*-analytic in* V.

Theorem 3.6. *For every analytic closed curve* γ *which does not divide* R, *there exists an* (F, G)*-analytic differential which has non zero* (F, G)*-period on* γ *and is everywhere regular.*

This is the immediate consequence of Theorem 3.1.

References

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