34. *Duality in the Linear Connections in the Large*

By Tsurusaburo TAKASU
Tohoku University, Sendai

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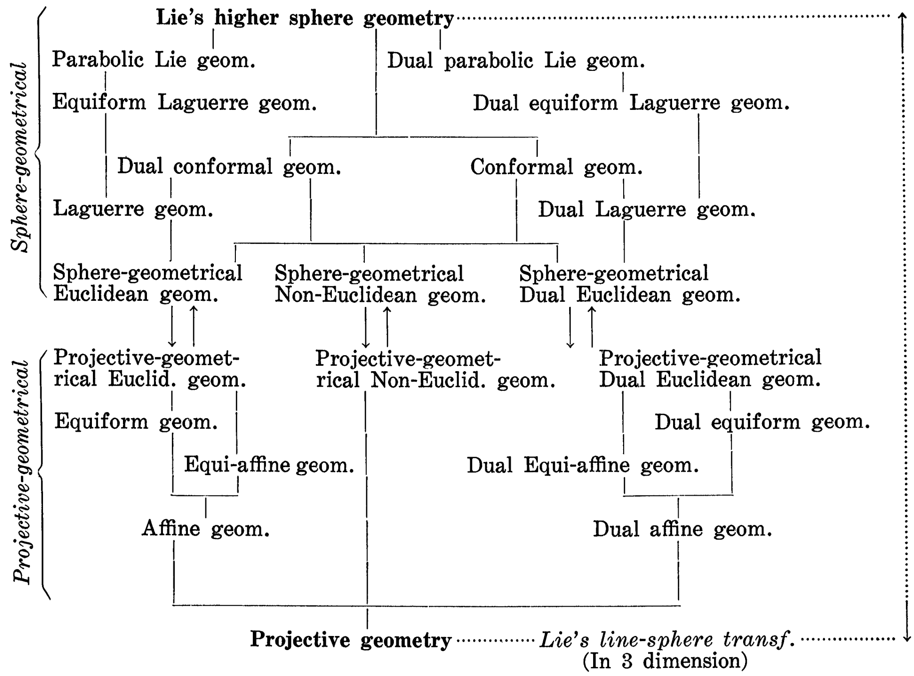
The present author has extended all the branches of the following table by extending respective group parameters to appropriate functions of coordinates, respective invariants being retained, and enabled us to realize all the geometries so extended in the differentiable manifolds ([1]-[14]):



He has established [10] a theory of linear connections in the large considering principal fibre bundles with structure groups extended in the manner as was stated above.

In this note it is aimed at to establish a duality in the linear connections in the large taking the case of the extended affine connections, it being possible to treat all the other cases similarly. A detailed exposition will be done in "The Tensor".

1. A System of Profitable Hypercomplex Units. For subsequent use, we will introduce a system of profitable hypercomplex units γ_p by the conditions:

$$(1.1)\qquad \gamma_p\gamma_q+\gamma_q\gamma_p=2\delta_{pq},\quad (p,q,\cdots=1,2,\cdots,n),$$

where δ_{pq} are Kronecker deltas. Then we have generally

$$(\gamma_pA^p)(\gamma_qB^q)=A^pB^p+\tfrac{1}{2}(\gamma_p\gamma_q)(A^pB^q-A^qB^p),\quad (p\neq q).$$

Let us set for this

$$(1.2)\qquad (\gamma_pA^p)(\gamma_qB^q)=A^pB^p+\mathfrak{D}A^p\wedge B^p,$$

the symbol $\mathfrak{D}$ indicating the dual part consisting of exterior products.

2. A Duality in the Equations of Structure. The equations of structure are fundamental for linear connections and the following theorem for them is established:

Theorem. *In order that n linearly independent linear differential forms*

$$(2.1)\qquad \omega^l=\omega^l_\mu(x^\nu)dx^\mu\quad (\lambda,\mu,\cdots,l,\cdots=1,2,\cdots,n)$$

and

$$(2.2)\qquad \theta^l_m=\Gamma^l_{nm}(x^\nu)\omega^n$$

defined on the $n^2-({}_nC_2+n)+n=\frac{n}{2}(n+1)$ *dimensional principal fibre bundle* $\mathfrak{B}$ *consisting of* $\{x^\lambda\}$ *and* $\{\omega^l_\mu\}$ *in the local may define an extended affine connection on a differentiable manifold M provided with local coordinate system* (x^λ), *it is necessary and sufficient that the "equations of structure"*

$$\begin{array}{ll|l}
(2.3) & d\omega^l+\theta^l_h\omega^h=\tfrac{1}{2}S^l_{mn}\omega^m\omega^n & \mathbf{d}\omega^l+\theta^l_h\wedge\omega^h=\tfrac{1}{2}T^l_{mn}\omega^m\wedge\omega^n \\
 & \quad =\omega^l_\lambda(d^2x^\lambda+\Gamma^\lambda_{\mu\nu}dx^\mu dx^\nu) & \quad =\omega^l_\lambda(d^2x^\lambda+\Gamma^\lambda_{\mu\nu}dx^\mu\wedge dx^\nu) \\
 & \quad =\omega^l_\lambda(d^2x^\lambda+\theta^\lambda_\mu dx^\mu), & \quad =\omega^l_\lambda(d^2x^\lambda+\theta^\lambda_\mu\wedge dx^\mu), \\
(2.3)' & d\theta^k_l+\theta^k_h\theta^h_l=\tfrac{1}{2}Q^k_{\cdot lmn}\omega^m\omega^n & \mathbf{d}\theta^k_l+\theta^k_h\wedge\theta^h_l=\tfrac{1}{2}R^k_{\cdot lmn}\omega^m\wedge\omega^n \\
 & \quad =\omega^k_\lambda\Omega^\mu_l\{(d\theta^\lambda_\mu+\theta^\lambda_\nu\theta^\nu_\mu) & \quad =\omega^k_\lambda\Omega^\mu_l\{(\mathbf{d}\theta^\lambda_\mu+\theta^\lambda_\nu\wedge\theta^\nu_\mu) \\
 & \qquad -(d\Theta^\lambda_\mu+\Theta^\lambda_\nu\Theta^\nu_\mu)\} & \qquad -(\mathbf{d}\Theta^\lambda_\mu+\Theta^\lambda_\nu\wedge\Theta^\nu_\mu)\} \\
 & \quad \equiv\omega^k_\lambda\Omega^\mu_l(\tfrac{1}{2}Q^\lambda_{\cdot\mu\alpha\beta} & \quad \equiv\omega^k_\lambda\Omega^\mu_l(\tfrac{1}{2}R^\lambda_{\cdot\mu\alpha\beta} \\
 & \qquad -\tfrac{1}{2}\overline{Q}^\lambda_{\cdot\mu\alpha\beta})dx^\alpha dx^\beta & \qquad -\tfrac{1}{2}\overline{R}^\lambda_{\cdot\mu\alpha\beta})dx^\alpha\wedge dx^\beta
\end{array}$$

are satisfied, where **d** *indicates the exterior differentials and*

$$(2.4)\qquad \Omega^\lambda_l\omega^l_\mu=\delta^\lambda_\mu \rightleftarrows \Omega^\lambda_m\omega^l_\lambda=\delta^l_m,$$

$$\begin{array}{ll|l}
(2.5) & S^\lambda_{\mu\nu}=\Gamma^\lambda_{\nu\mu}+\Gamma^\lambda_{\mu\nu}, & T^\lambda_{\mu\nu}=\Gamma^\lambda_{\nu\mu}-\Gamma^\lambda_{\mu\nu}, \\
(2.6) & S^l_{mn}=\omega^l_\lambda\Omega^\mu_m\Omega^\nu_nS^\lambda_{\mu\nu}, & T^l_{mn}=\omega^l_\lambda\Omega^\mu_m\Omega^\nu_nT^\lambda_{\mu\nu}, \\
(2.7) & Q^k_{\cdot lmn}=\omega^k_\sigma\Omega^\lambda_l\Omega^\mu_m\Omega^\nu_nQ^\sigma_{\cdot\lambda\mu\nu}, & R^k_{\cdot lmn}=\omega^k_\sigma\Omega^\lambda_l\Omega^\mu_m\Omega^\nu_nR^\sigma_{\cdot\lambda\mu\nu}, \\
(2.8) & Q^\sigma_{\cdot\lambda\mu\nu}=\dfrac{\partial\Gamma^\sigma_{\lambda\nu}}{\partial x^\mu}+\dfrac{\partial\Gamma^\sigma_{\lambda\mu}}{\partial x^\nu} & R^\sigma_{\cdot\lambda\mu\nu}=\dfrac{\partial\Gamma^\sigma_{\lambda\nu}}{\partial x^\mu}-\dfrac{\partial\Gamma^\sigma_{\lambda\mu}}{\partial x^\nu} \\
 & \qquad +(\Gamma^\kappa_{\lambda\mu}\Gamma^\sigma_{\kappa\nu}+\Gamma^\kappa_{\lambda\nu}\Gamma^\sigma_{\kappa\mu}), & \qquad -(\Gamma^\kappa_{\lambda\mu}\Gamma^\sigma_{\kappa\nu}-\Gamma^\kappa_{\lambda\nu}\Gamma^\sigma_{\kappa\mu}),
\end{array}$$

$$(2.9)\qquad \Lambda^\lambda_{\mu\nu}=\Omega^\lambda_l\frac{\partial\omega^l_\mu}{\partial x^\nu}\equiv-\omega^l_\mu\frac{\partial\Omega^\lambda_l}{\partial x^\nu},\quad \Lambda^l_{mn}=\omega^l_\lambda\frac{\partial\Omega^\lambda_m}{\omega^n}\equiv-\Omega^\lambda_m\frac{\partial\omega^l_\lambda}{\omega^n},$$

$$(2.10)\qquad \Theta^\lambda_\mu=\Lambda^\lambda_{\nu\mu}dx^\nu.$$

The proof will be executed parallelly to that of [10] taking the following left-hand side in place of the following right-hand side:

$$
\begin{array}{l|l}
\gamma_p\omega^p=\gamma_p(\omega^p_\sigma dx^\sigma) & \omega^p=\omega^p_\sigma dx^\sigma \\
(\gamma_q\Gamma^h_{kq})(\gamma_p\omega^p) & \Gamma^h_{pk}\omega^p \\
(\gamma_q\Omega^\lambda_q)(\gamma_p\theta^p_l) & \Omega^\lambda_p\theta^p_l
\end{array}
$$

etc.,

starting with the identity

(2.11) $$\Omega^\lambda_p\theta^p_l=\Omega^\mu_l(\theta^\lambda_\mu-\Theta^\lambda_\mu),$$

which is obtained by straight forward calculation. The form Θ^λ_μ is that of the teleparallelism $\Lambda^\lambda_{\mu\nu}$ for ω^l_μ and Ω^λ_l.

The two sides of (2.3), (2.3)′ are obtained unifiedly in the form:

(the left-hand side)$+\mathfrak{D}$ (the right-hand side).

It is remarkable that the two sides of the equations of structure (2.3), (2.3)′ are of the same weight.

The local laws indicated by the Greek indices in U_α [10] are mapped *continuingly* (by the so-called paste condition for the differentiable manifolds) onto the atlas $\bigcup_\alpha U_\alpha$.

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