49. On Cesàro Summability of Fourier-Laguerre Series

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(Comm. by Kinjirô KUNUGI, M.J.A., April 12, 1963)

1. The Fourier-Laguerre expansion corresponding to a function $f(x)\in L(0,\infty)$ is given by

$$f(x)\sim\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x) \tag{1.1}$$

where

$$\Gamma(\alpha+1)\binom{n+\alpha}{n}a_n=\int_0^{+\infty}e^{-x}x^{\alpha}f(x)L_n^{(\alpha)}(x)dx, \tag{1.2}$$

and $L_n^{(\alpha)}$ denotes the Laguerre polynomial of order α.

At the point $X=0$

$$\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(0)=\frac{1}{\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\int_0^{\infty}e^{-t}t^{\alpha}f(t)L_n^{(\alpha)}(t)dt. \tag{1.3}$$

Denoting the Cesàro means of order k of the series (1.1) at the point $X=0$ by $\sigma_n^k(0)$, we easily have

$$\sigma_n^k(0)=\{A_n^{(k)}\Gamma(\alpha+1)\}^{-1}\int_0^{\infty}e^{-t}t^{\alpha}f(t)L_n^{(\alpha+k+1)}(t)dt. \tag{1.4}$$

Szegö [1] has studied the (C,k) summability of Laguerre series corresponding to a continuous function for $k>\alpha+\frac{1}{2}$.

In the present paper I prove the following more general theorem:–

Theorem. If $f(x)$ be integrable in $(0,\infty)$ and if it satisfies the following conditions

$$\int_1^{\infty}e^{-\frac{x}{2}}x^{\alpha-k-\frac{1}{3}}|f(x)|dx<\infty, \text{ and} \tag{1.5}$$

$$\int_0^{t}|f(t)|dt=o(t), \tag{1.6}$$

then the Laguerre series of $f(x)$ is (C,k) summable at $x=0$ with the sum of $f(0)$ provided that $k>\alpha+\frac{1}{2}$.

2. We shall take help of the following lemmas in the proof of the theorem:—

Lemma 1 (Szegö [2], p. 172). Let α be arbitrary and real, C and ω fixed positive constants, and let $n\to\infty$. Then

$$L_n^{(\alpha)}(x)=\begin{cases} x^{-\frac{\alpha}{2}-\frac{1}{4}}0(n^{\frac{\alpha}{2}-\frac{1}{4}}), & \text{if, } \frac{c}{n}\le x\le\omega, \\ 0(n^{\alpha}) & \text{if, } 0\le x\le\frac{c}{n}, \end{cases} \tag{2.1}$$

Lemma 2 (Szegö [2], p. 235). Let α and λ be arbitrary and real

$a>0$, $0<\eta<4$. Then for $n\to\infty$

$$\max e^{-\frac{x}{2}}x^{\lambda}\,|L_n^{(\alpha)}(x)|\sim n^Q \tag{2.2}$$

where

$$Q=\begin{cases}\max\left(\lambda-\dfrac{1}{2},\dfrac{\alpha}{2}-\dfrac{1}{4}\right), \text{ if } a\le x\le(4-\eta)n\\ \max\left(\lambda-\dfrac{1}{3},\dfrac{\alpha}{2}-\dfrac{1}{4}\right), \text{ if } x\ge a.\end{cases} \tag{2.3}$$

3. *Proof of the theorem.* Without loss of generality we may take $f(0)=0$. Now

$$\begin{aligned}\sigma_n^{(k)}(0)&=\frac{1}{\Gamma(\alpha+1)A_n^{(k)}}\int_0^\infty e^{-t}t^\alpha f(t)L_n^{(\alpha+k+1)}(t)dt.\\ &=\frac{1}{\Gamma(\alpha+1)A_n^{(k)}}\left[\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{S}+\int_S^\infty\right]\\ &=\frac{1}{\Gamma(\alpha+1)A_n^{(k)}}[I_1+I_2+I_3], \text{ say.}\end{aligned}$$

$$\begin{aligned}|I_1|&\le\int_0^{\frac{1}{n}}e^{-t}t^\alpha|f(t)|\,|L_n^{(\alpha+k+1)}(t)|\,dt\\ &=0(n^{\alpha+k+1})\int_0^{\frac{1}{n}}e^{-t}t^\alpha|f(t)|\,dt, \text{ by Lemma 1.}\\ &=0(n^{\alpha+k+1})\frac{1}{n^\alpha}\int_0^{\frac{1}{n}}|f(t)|\,dt.\\ &=o(n^k), \text{ from (1.6)} \qquad (3.1)\end{aligned}$$

$$\begin{aligned}|I_2|&=\left|\int_{\frac{1}{n}}^{S}e^{-t}t^\alpha f(t)L_n^{(\alpha+k+1)}(t)dt\right|\\ &=0\left(\int_{\frac{1}{n}}^{S}e^{-t}t^\alpha|f(t)|\,t^{-\frac{\alpha+k+1}{2}-\frac{1}{4}}n^{\frac{\alpha+k+1}{2}-\frac{1}{4}}dt\right) \quad \text{By Lemma 1.}\\ &=0(n^{+\frac{\alpha}{2}+\frac{k}{2}+\frac{1}{4}})\int_{\frac{1}{n}}^{S}|f(t)|\,t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{3}{4}}dt. \qquad (3.2)\end{aligned}$$

On integrating by parts we find on putting

$\int_0^t|f(u)|\,du=F(t)$ that

$$\begin{aligned}\int_{\frac{1}{n}}^{S}|f(t)|\,t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{3}{4}}dt&=[F(t)t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{3}{4}}]_{\frac{1}{n}}^{S}-\int_{\frac{1}{n}}^{S}F(t)\,t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{7}{4}}dt\\ &=F(S)S^{\frac{\alpha}{2}-\frac{k}{2}-\frac{3}{4}}+F\left(\frac{1}{n}\right)n^{-\frac{\alpha}{2}+\frac{k}{2}+\frac{3}{4}}\\ &\quad-\int_{\frac{1}{n}}^{S}0(t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{3}{4}})dt\\ &=0(1)+0(n^{-\frac{\alpha}{2}+\frac{k}{2}-\frac{1}{4}})-0(1)+0(n^{-\frac{\alpha}{2}+\frac{k}{2}-\frac{1}{4}}).\end{aligned}$$

Therefore (3.2) give that

$$|I_2|=0(n^k). \tag{3.3}$$

Coming now to I_3, we make use of Lemma 2. Replacing α by $\alpha+k+1$ and putting $\lambda=k+\frac{1}{3}$, we see that $\lambda-\frac{1}{3}=k>\frac{\alpha+k+1}{2}-\frac{1}{4}$ since $k>\alpha+\frac{1}{2}$. So,

$$|I_3|=0(1)\int_S^\infty e^{-\frac{t}{2}}t^\alpha|f(t)|t^{-k-\frac{1}{3}}n^k\,dt$$

$$=0(n^k)\int_S^\infty e^{-\frac{t}{2}}t^{\alpha-k-\frac{1}{3}}|f(t)|\,dt$$

$$=o(n^k), \tag{3.4}$$

because we can choose S as large as we please.

Combining (3.1), (3.3), and (3.4), we have the required result.

References

[1] Szegö, G.: Beiträge zur Theorie der Laguerreschen Polynome., I. Entwicklungs-satze, Mathematische Zeitschrift, **25**, 87–115 (1926).

[2] Szegö, G.: Orthogonal polynomials, American Math. Soc. Colloquium Publications, Vol. XXIII (1939).