62. A Note on the Completion Theory

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In this paper, a generalization of the theory of completion due to Dr. Cohen ([3] [4] see also [6]) is attempted.

It is well known that the constructive method of Cantor-Hausdorff for the completion of metric spaces can be generalized to the case of uniform spaces (see for example [2] and [9]).

In recent years, G. Areskin gave a theory free from the uniformity, namely, which is valid at least for the general regular spaces [1], and P. Holm treated the case when the space has a system of finite coverings. In this paper, we shall consider the case when this system of covering is not necessarily finite. We have already given the notion of the Cauchy filter for the spaces with arbitrary coverings [9]. Moreover, it can be shown by some easy examples that our method is applicable naturally to non-regular spaces.

An *extention* X' of topological space X, is such a topological space that $X\subseteq X'$ and $\overline{X}=X'$. A completion X^* of a topological space is such an extention that; there exists a completion of a topological space X having some quasi-topological property P and if X consists of all the rational numbers and its some property P then X^* is the set of all real number; X^* is uniquely determined, for X and its some property P; and etc.$\cdots$. The completion theory is the theory of these quasi-topological property P.

At first we must define the completion, when we deal with considerably general property P.

In uniform space, even in Dr. Cohen's case, P is a global property for a covering system, but in this paper, Theorem 1 shows P is local one for filters which dosen't converge any points.

A *filter base* $\mathfrak{f}$ in a set X is a family of non-void subsets of X such that, for any $A, B\in\mathfrak{f}$, $C\subseteq A\frown B$ for some $C\in\mathfrak{f}$.

A *filter* $\mathfrak{f}$ is a set X is a filter base such that, if $A\in\mathfrak{f}$ and $A\subseteq B$ then $B\in\mathfrak{f}$. Let $\mathfrak{f}$ be a filter base, then $\{A\mid A\supseteq C, C\in\mathfrak{f}\}$ is a filter, and it is said to be *generated* by $\mathfrak{f}$. When a set X is contained in another set X^*, any filter $\mathfrak{f}$ in X generates a filter $\mathfrak{f}^*$ in X^*.

In a topological space X, a filter base $\mathfrak{f}$ is said to be *convergent* to a point x of X if the neighborhood system of x is contained in the filter which is generated by $\mathfrak{f}$.

A bow $\mathfrak{B}$ of X is a base of open sets of X which has a certain index set Λ, and

(i) to any $\alpha \in \Lambda$, there corresponds a covering $\mathfrak{V}(\alpha)$ of X,

(ii) $\mathfrak{V} = U_{\alpha \in \Lambda} \mathfrak{V}(\alpha)$.

Combining the topological space X with its a bow $\mathfrak{V}$, we shall denote X as $(X; \mathfrak{V}, \Lambda)$ or simply $(X; \mathfrak{V})$, and let's call it by a *bow space.*

A filter $\mathfrak{f}$ is a Cauchy filter if for any $\alpha \in \Lambda$, $\mathfrak{V}(\alpha) \frown \mathfrak{f} \neq \phi$.

A *leg* in a bow space $(X; \mathfrak{V})$ is a Cauchy filter in X which does not converge to any point of X. When $(X; \mathfrak{V})$ has no leg, $(X; \mathfrak{V})$ is a *complete* space.

A bow space $(X^*; \mathfrak{V}^*)$ is a *completion* of $(X; \mathfrak{V})$, when bows $\mathfrak{V}$ and $\mathfrak{V}^*$ have same index set Λ and the followings hold;

(1) $X \subseteq X^*$

(2) $(X^*; \mathfrak{V}^*)$ is complete,

(3) for any $\alpha \in \Lambda$, there exists a one valued mapping from $\mathfrak{V}(\alpha)$ onto $\mathfrak{V}^*(\alpha)$

$$V \leftrightarrow V^* \qquad (V \in \mathfrak{V}(\alpha),\ V^* \in \mathfrak{V}^*(\alpha))$$

such that, for any $V \in \mathfrak{V}$, $V^* \frown X = V$,

(4) let $x \in X^* \sim X$ and $V \in \mathfrak{V}(\alpha)$, and if every leg in $(X; \mathfrak{V})$, converging to x in X^* always contains V, then $x \in V^*$,

(5) any leg in $(X; \mathfrak{V})$ converges to at most one point in X^*, and for any point of $X^* \sim X$, there exists at least one leg which converges to the point.

Then the $*$ dosen't depend on α.

Assume that, for a bow space $(X; \mathfrak{V})$, there exists a completion $(X^*; \mathfrak{V}^*)$. Let $\mathfrak{f}$ be any leg in $(X; \mathfrak{V})$. And let's denote by $[\mathfrak{f}]$ the intersection of all legs in $(X; \mathfrak{V})$, contained in $\mathfrak{f}$. It is even a leg. A member of $\mathfrak{V} \frown [\mathfrak{f}]$ is a *body* of $\mathfrak{f}$. And a member of $\mathfrak{V}(\alpha) \frown [\mathfrak{f}]$ is an *α-body* of $\mathfrak{f}$. Therefore, for any leg $\mathfrak{f}$ in $(X; \mathfrak{V})$ and for any $\alpha \in \Lambda$, there exists an α-body of $\mathfrak{f}$. A leg $\mathfrak{f}$ is a *minimal leg* if $\mathfrak{f} = [\mathfrak{f}]$.

THEOREM 1. *A bow space $(X; \mathfrak{V})$ has its completion if and only if the next conditions hold;*

1. *if $\mathfrak{f}$ is a leg in $(X; \mathfrak{V})$, $[\mathfrak{f}]$ is also a leg,*

2. *for any leg $\mathfrak{f}$ in $(X; \mathfrak{V})$ and for any two bodies V_1, V_2 of $\mathfrak{f}$, there exists a body V_3 of $\mathfrak{f}$ such that $V_3 \subseteq V_1 \frown V_2$.*

THEOREM 2. *A completion of $(X; \mathfrak{V})$ is uniquely determined by its bow $\mathfrak{V}$.*

THEOREM 3. *Assume that $(X; \mathfrak{V}, \Lambda)$ and $(X; \mathfrak{W}, \Omega)$ have a same topology of X, and that there exist completions $(X^*; \mathfrak{V}^*, \Lambda)$, $(X^+; \mathfrak{W}^+, \Omega)$ of $(X; \mathfrak{V}, \Lambda)$, $(X; \mathfrak{W}, \Omega)$ respectively. Then there exists a homeomorphic mapping f from $(X^*; \mathfrak{V}^*, \Lambda)$ onto $(X^+; \mathfrak{W}^+, \Omega)$, such that $f(x) = x$ for any $x \in X$, if and only if every leg in $(X; \mathfrak{V}, \Lambda)$ is also a leg in $(X, \mathfrak{W}, \Omega)$, and conversely.*

Further we get a following useful theorem.

THEOREM 4. *Let f be continuous mapping from a bow space $(X;\mathfrak{B})$, which has its completion $(X^*;\mathfrak{B}^*)$, into a topological space Y such that for any $y\in Y$, and for its every neighborhood W, there exists a neighborhood V of y and $\bar{V}\subseteq W$. Then, there exists a continuous mapping F from the completion $(X^*;\mathfrak{B}^*)$ of $(X;\mathfrak{B})$ into Y such that for any $x\in X$, $F(x)=f(x)$, if and only if for any leg $\mathfrak{f}$ in $(X;\mathfrak{B})$, the filter base $\{f(A)\,|\,A\in\mathfrak{f}\}$ converges to a point in Y.*

Next, we consider about the product of bow spaces.

Let $(X_\lambda;\mathfrak{B}_\lambda, \Lambda_\lambda)$ be a bow space for every $\lambda\in\Delta$, but if $\lambda\neq\mu$ then $\Lambda_\lambda\frown\Lambda_\mu=\phi$, for every $\lambda,\mu\in\Delta$, then the *product space* $(X;\mathfrak{B},\Lambda)$ of them is such a bow space that;

(I) $X=\prod_{\lambda\in\Delta}X_\lambda$ and X has the weak topology,

(II) Λ is the family consisting of all such finite subsets of $U_{\lambda\in\Delta}\Lambda_\lambda$ that never contain two elements of same Λ_λ, and for every $\alpha\in\Lambda$,

$$\mathfrak{B}(\alpha)=\left\{\prod_{\lambda\in\Delta}K_\lambda\,\middle|\,\begin{array}{l}\alpha\frown\Lambda_\lambda\ni\alpha_\lambda\longrightarrow K_\lambda\in\mathfrak{B}_\lambda(\alpha_\lambda)\\ \alpha\frown\Lambda_\lambda=\phi\longrightarrow K_\lambda=X_\lambda\end{array}\right\},$$

(III) $\mathfrak{B}=U_{\alpha\in\Delta}\mathfrak{B}(\alpha)$.

Let's denote the product spaces of $(X_\lambda;\mathfrak{B}_\lambda,\Lambda_\lambda)$, by $\prod_{\lambda\in\Delta}(X_\lambda;\mathfrak{B}_\lambda,\Lambda_\lambda)$, and denote, by P_λ, the projection of $X=\prod X_\lambda$ onto its λ-component X_λ.

THEOREM 5. *A product space $\prod(X_\lambda.\mathfrak{B}_\lambda,\Lambda_\lambda)$ of bow spaces $(X_\lambda;\mathfrak{B}_\lambda,\Lambda_\lambda)$ is complete if and only if each $(X_\lambda;\mathfrak{B}_\lambda,\Lambda_\lambda)$ is complete.*

THEOREM 6. *If T_2 bow spaces $(X_\lambda;\mathfrak{B}_\lambda,\Lambda_\lambda)$ have their completions $(X_\lambda^*;\mathfrak{B}_\lambda^*,\Lambda_\lambda)$ respectively and they are also T_2 spaces, then the product space $\prod(X_\lambda;\mathfrak{B}_\lambda,\Lambda_\lambda)$ has its completion $\prod(X_\lambda^*;\mathfrak{B}_\lambda^*,\Lambda_\lambda)$.*

We call a Cauchy filter $\mathfrak{f}$ in a space $(X;\mathfrak{B})$ a *minimal Cauchy filter* in X, if and only if there is no Cauchy filter, in X, which is properly contained in $\mathfrak{f}$.

PROPOSITION. *A completion $(X^*;\mathfrak{B}^*)$ of a T_2 bow space $(X;\mathfrak{B})$ is a T_2 space if and only if;*

(H_1) *for any leg $\mathfrak{f}$ in X and any point $x\in X$, there exist some body V of $\mathfrak{f}$, and some neighborhood W of x, such that $V\frown W=\phi$,*

(H_2) *for any legs $\mathfrak{f}$ and $\mathfrak{g}$ in $(X;\mathfrak{B})$, if $\mathfrak{f}\frown\mathfrak{g}$ is not a Cauchy filter, then there exists some bodies V and W of $\mathfrak{f}$ and $\mathfrak{g}$, respectively, and $V\frown W=\phi$.*

References

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