59. On Bochner Transforms

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1. If $\varphi(r)$ is a function of the distance r from the origin of h-dimensional Euclidean space, then the Fourier transform of $\varphi(r)$, that is the integral transform by the kernel function $\exp(2\pi_i(x_1y_1+ +\cdots+x_hy_h))$, is also a function of r and is expressed as following:

$$T\varphi(r)=2\pi r^{1-\frac{h}{2}}\int_0^\infty J_{\frac{h}{2}-1}(2\pi r\rho)\rho^{\frac{h}{2}}\varphi(\rho)\,d\rho,$$

where $J_\nu(x)$ is Bessel function ([1] p. 69, Theorem 40).

By the general theory of Fourier transform the linear operator T has the properties:

(a) *T transforms $\varphi(ur)$ to $|u|^{-h}T\varphi\left(\frac{r}{u}\right)$ if $u\neq 0$,*

(b) *T transforms $e^{-\pi r^2}$ to $e^{-\pi r^2}$,*

(c) *there exists a number series $\{a_0, a_1,\cdots\}$ which satisfies*

$$\sum_{n=0}^\infty a_n\varphi(\sqrt{n})=\sum_{n=0}^\infty a_nT\varphi(\sqrt{n}) \text{ (Poisson summation formula),}$$

(d) $T^2\varphi=\varphi$ (the inversion formula),

and

(e) $\int_0^\infty |T\varphi(r)|^2r^{k-1}\,dr=\int_0^\infty |\varphi(r)|^2r^{k-1}\,dr$ (Parseval formula).

In his paper [2] Bochner proved that the properties (a), (b) and (c) characterize the operator T.

We shall describe here the theorem of Bochner in somewhat modified form:

Let us denote by $\mathfrak{P}_0$ the family of all functions $\varphi(x)$ on $[0, \infty)$ such that $\left(\frac{d}{xdx}\right)^r\varphi(x)$ exists at 0 for any r and every derivative of $\varphi(x)$ decreases rapidly as x tends to infinity.

Theorem of Bochner. *Let T be a linear operator from $\mathfrak{P}_0$ to $\mathfrak{P}_0$ which satisfies the following conditions:*

(A) *T transforms $\varphi(ux)$ to $|u|^{-h}T\varphi\left(\frac{x}{u}\right)$ for any $u\neq 0$, where h is a positive constant,*

(B) *$e^{-\frac{2\pi}{\lambda}x^2}$ is an eigenfunction of T, where λ is a positive constant, and*

(C) *there exists number series $\{a_0, a_1, a_2,\cdots\}$ such that $\sum_{n=1}^\infty \frac{|a_n|}{n^{s_0}}$ converges for a positive number s_0 and*

$$\sum_{n=0}^{\infty} a_n \varphi(\sqrt{n}) = \sum_{n=0}^{\infty} a_n T\varphi(\sqrt{n})$$

for any $\varphi(x)$ *in* $\mathfrak{P}_0$, *then* $T\varphi(x)$ *is equal to the* Bochner transform

$$T_{\lambda,h}\varphi(x) = \frac{4\pi}{\lambda} x^{-\frac{h}{2}+1} \int_0^{\infty} J_{\frac{h}{2}-1}\left(\frac{4\pi}{\lambda} xt\right) t^{\frac{h}{2}} \varphi(t)\, dt$$

of $\varphi(x)$ *or equal to* $-T_{\lambda,h}\varphi(x)$, *and the Mellin transform of* $T_{\lambda,h}\varphi(x)$ *is* $\Phi(h-s)\left(\frac{2\pi}{\lambda}\right)^{-s+\frac{h}{2}} \Gamma\left(\frac{s}{2}\right) \Big/ \Gamma\left(\frac{h-s}{2}\right)$ *where* $\Phi(s)$ *is the Mellin transform of* $\varphi(x)$.

At this opportunity it may be remarked that the result of our previous paper [3] is a direct consequence of Bochner's theory.

In the following we shall investigate the properties of $T_{\lambda,k}$ and give a generalization of the central limit theorem in the probability theory (Proposition 9).

2. Using Bochner's theorem it is easily proved that the Mellin transform of $T^2\varphi$ is equal to $\Phi(s)$. Therefore we get

Proposition 1. $T_{\lambda,k}(T_{\lambda,k}\varphi) = \varphi$.

Moreover an analogy of Parseval formula can be stated as follows:

Proposition 2. $\int_0^{\infty} x^{h-1} |T_{\lambda,h}\varphi(x)|^2\, dx = \int_0^{\infty} x^{h-1} |\varphi(x)|^2\, dx.$

Proof. We have

$$\int_0^{\infty} x^{h-1} |T_{\lambda,h}\varphi(x)|^2\, dx$$

$$= \left(\frac{4\pi}{\lambda}\right)^2 \int_0^{\infty} x dx \int_0^{\infty} J_{\frac{h}{2}-1}\left(\frac{4\pi}{\lambda} xt\right) t^{\frac{h}{2}} \varphi(t)\, dt \cdot \int_0^{\infty} J_{\frac{h}{2}-1}\left(\frac{4\pi}{\lambda} xu\right) u^{\frac{h}{2}} \overline{\varphi(u)}\, du$$

$$= \int_0^{\infty} u^{\frac{h}{2}} \overline{\varphi(u)}\, du \int_0^{\infty} J_{\frac{h}{2}-1}(xu)\, x dx \int_0^{\infty} J_{\frac{h}{2}-1}(xt) t (t^{\frac{h}{2}-1} \varphi(t)) dt.$$

By Hankel's inversion formula (Titchmarsh [4] p. 241, Watson [5], p. 456 (1)) this is equal to

$$\int_0^{\infty} u^{\frac{h}{2}} \overline{\varphi(u)} \cdot u^{\frac{h}{2}-1} \varphi(u)\, du.$$

Corollary. $\int_0^{\infty} x^{h-1} T_{\lambda,h}\varphi(x) T_{\lambda,h}\psi(x)\, dx = \int_0^{\infty} x^{h-1} \varphi(x)\psi(x)\, dx.$

Now we shall give a relation between $T_{\lambda,h}$ and $T_{\lambda,h+2}$. Because $\frac{d}{dx}(x^{\nu} J_{\nu}(x)) = x^{\nu} J_{\nu-1}(x)$ we have, integrating by parts,

$$T_{\lambda,h}\varphi(x) = -x^{-\frac{h}{2}} \int_0^{\infty} J_{\frac{h}{2}}\left(\frac{4\pi}{\lambda} xt\right) t^{\frac{h}{2}} \varphi'(t)\, dt.$$

Thus we have

Proposition 3. *For* $h>0$, $T_{\lambda,h}f(x)$ *is equal to* $-\frac{\lambda}{4\pi} T_{\lambda,h+2}\left\{\frac{1}{x} f'(x)\right\}(x)$ *if* $\frac{1}{x} f'(x)$ *is defined at* $[0, \infty)$.

Then, we estimate the value of $T_{\lambda,h}f(x)$:

$$T_{\lambda,h}f(x)=-\frac{\lambda}{4\pi}T_{\lambda,h+2}\left\{\frac{d}{xdx}f(x)\right\} \quad \text{(Proposition 3)}$$
$$=\left(-\frac{\lambda}{4\pi}\right)^r T_{\lambda,h+2r}\left\{\left(\frac{d}{xdx}\right)^r f(x)\right\}$$
$$=O\left(x^{-\left(\frac{h}{2}+r-1\right)-\frac{1}{2}}\int_0^\infty t^{\frac{h}{2}+r-\frac{1}{2}}\left|\left(\frac{d}{tdt}\right)^r f(t)\right|dt\right),$$

because $J_p(x)=O(x^{-\frac{1}{2}})$.

Proposition 4. *Let h be a positive number, r be a non-negative integer and $f(x)$ be a function such that*

$$\int_0^\infty\left|\left(\frac{d}{tdt}\right)^r f(t)\right|t^{\frac{h}{2}+r-\frac{1}{2}}dt \text{ exists. Then}$$
$$T_{\lambda,h}f(x)=O\left(x^{-\frac{h}{2}-r+\frac{1}{2}}\int_0^\infty\left|\left(\frac{d}{tdt}\right)^r f(t)\right|t^{\frac{h}{2}+r-\frac{1}{2}}dt\right).$$

Using the identity

$$\frac{d}{dx}(x^{-\nu}J_\nu(x))=-x^{-\nu}J_{\nu+1}(x)$$

we can calculate the derivative of $T_{\lambda,h}\varphi(x)$. Namely,

$$\frac{d}{dx}(T_{\lambda,h}\varphi(x))=\frac{d}{dx}\left(\frac{4\pi}{\lambda}\int_0^\infty x^{-\frac{h}{2}+1}J_{\frac{h}{2}-1}\left(\frac{4\pi}{\lambda}xt\right)t^{\frac{h}{2}}\varphi(t)dt\right)$$
$$=\frac{4\pi}{\lambda}\int_0^\infty\left(-\frac{4\pi}{\lambda}tx^{-\frac{h}{2}-1}J_{\frac{h}{2}}\left(\frac{4\pi}{\lambda}xt\right)\right)t^{\frac{h}{2}}\varphi(t)\,dt$$
$$=-\frac{4\pi}{\lambda}xT_{\lambda,h+2}\varphi(x).$$

Proposition 5. $\dfrac{d}{dx}(T_{\lambda,h}\varphi(x))=-\dfrac{4\pi}{\lambda}xT_{\lambda,h+2}\varphi(x)$.

The existence of the following proposition is the reason why we select in (B) the special eigenfunction $e^{-\frac{2\pi}{\lambda}x^2}$.

Proposition 6. *If $\varphi(x)$ is an eigenfunction with respect to $T_{\lambda,h}$ and $T_{\lambda,h+2}$ belonging to $\mathfrak{P}_0$, then $\varphi(x)$ is of the form $ce^{-\frac{2\pi}{\lambda}x^2}$.*

Proof. Let

$$T_{\lambda,h}\varphi(x)=\alpha\varphi(x) \text{ and } T_{\lambda,h+2}\varphi(x)=\beta\varphi(x).$$

Then α and β are equal to 1 or -1 by Proposition 1, and $\alpha\varphi'(x)=-\frac{4\pi}{\lambda}x\beta\varphi(x)$ by Proposition 5. (Q.E.D.)

Now we have to introduce a topology on the linear space $\mathfrak{P}_0$. We take as a basis of neighbourhoods of 0 the family of the sets of the following type:

$$V(n,m,\varepsilon)=\left\{\varphi\in\mathfrak{P}_0:\left|(1+x)^m\left(\frac{d}{xdx}\right)^j\varphi(x)\right|<\varepsilon \right.$$
$$\left.\text{for any } x\geq 0 \text{ and for } j=0,1,2,\cdots,n\right\},$$

where n, m are arbitrary non-negative integers and ε is an arbitrary positive number. Concerning the Theorem of Bochner, we shall add the following

Proposition 7. $(\varphi, \psi)\to\varphi\psi$ is a continuous linear mapping from $\mathfrak{P}_0\times\mathfrak{P}_0$ to $\mathfrak{P}_0$ and $\varphi\to T_{\lambda,h}\varphi$ is a continuous linear mapping from $\mathfrak{P}_0$ onto itself.

Proof. The continuity of $(\varphi, \psi)\to\varphi\psi$ is a direct consequence of the definition of our topology. By Proposition 5 it is enough to prove the remainder of the proposition if we show for any given g, l and ε there exists a neighbourhood of 0 such that $|(1+x)^l T_{\lambda,g}\varphi(x)|<\varepsilon$ for all $x\geq 0$. In the case $x\geq 1$ this is proved by Proposition 4 and in the case $0\leq x\leq 1$ by the continuity of $x^{-\nu}J_\nu(x)$.

4. For two functions $\varphi(x_1,\cdots,x_h)$ and $\psi(x_1,\cdots,x_h)$ in h-variables the convolution of φ and ψ is defined by the formula

$$\varphi*\psi(x_1,\cdots,x_h)=\int\cdots\int\varphi(x_1-t_1,\cdots,x_h-t_h)\psi(t_1,\cdots,t_h)\,dt_1\cdots dt_h.$$

If φ and ψ depend only on $x=\sqrt{x_1^2+\cdots+x_n^2}$, then $\varphi*\psi$ is also a function in x only. And it is well known, if we regard $\varphi(x)$ and $\psi(x)$ as functions in x,

$$T_{2,h}(\varphi*\psi)=T_{2,h}\varphi\cdot T_{2,h}\psi.$$

Now we shall define the (λ, h) convolution $*_{\lambda,h}$ for any pair of positive numbers λ and h using the operator $T_{\lambda,h}$, that is,

$$\varphi*_{\lambda,h}\psi=T_{\lambda,h}(T_{\lambda,h}\varphi\cdot T_{\lambda,h}\psi).$$

Proposition 8. *(λ, h) convolution has the properties:*

(i) *if $\lambda=2$ and h is a natural number then $*_{\lambda,h}$ is identical with ordinary h-dimensional convolution,*

(ii) $T_{\lambda,h}(\varphi*_{\lambda,h}\psi)=T_{\lambda,h}\varphi\cdot T_{\lambda,h}\psi$,

(iii) *it is associative and commutative,*

(iv) *$(\varphi, \psi)\to\varphi*_{\lambda,h}\psi$ is a continuous bilinear mapping $\mathfrak{P}_0\times\mathfrak{P}_0$ to $\mathfrak{P}_0$, and*

(v) $$\varphi*_{\lambda,h}\psi(0)=\frac{1}{\Gamma\left(\frac{h}{2}\right)}\left(\frac{2\pi}{\lambda}\right)^{\frac{h}{2}}\int_0^\infty t^{h-1}\varphi(t)\psi(t)\,dt.$$

Proof. We get (i), (iii) directly from the definition of $*_{\lambda,h}$, (ii) by Proposition 1 and (iv) is a consequence of Proposition 7.

Finally we shall prove (v). By the Theorem of Bochner we have

$$T_{\lambda,h}\varphi(0)=\left(\frac{2\pi}{\lambda}\right)^{\frac{h}{2}}\frac{1}{\Gamma\left(\frac{h}{2}\right)}\int_0^\infty t^{h-1}\varphi(t)\,dt \text{ for any } \varphi \text{ in } \mathfrak{P}_0.$$

Therefore

$$\varphi*_{\lambda,h}\psi(0)=\left(\frac{2\pi}{\lambda}\right)^{\frac{h}{2}}\frac{1}{\Gamma\left(\frac{h}{2}\right)}\int_0^\infty t^{h-1}\,T_{\lambda,h}\varphi(t)\,T_{\lambda,h}\psi(t)\,dt$$

$$=\left(\frac{2\pi}{\lambda}\right)^{\frac{h}{2}}\frac{1}{\Gamma\left(\frac{h}{2}\right)}\int_0^\infty t^{h-1}\varphi(t)\psi(t)\,dt$$

by Corollary of Proposition 2. (Q.E.D.)

Now we shall give an analogy of the central limit theorem in the probability theory.

Proposition 9. *Put* $\alpha=\frac{1}{\Gamma\left(\frac{h}{2}\right)}\left(\frac{2\pi}{\lambda}\right)^{\frac{h}{2}}\int_0^\infty x^{h-1}\varphi(x)\,dx$ *and* $\beta=\frac{1}{\Gamma\left(\frac{h}{2}+1\right)}\left(\frac{2\pi}{\lambda}\right)^{\frac{h}{2}+1}\int_0^\infty x^{h+1}\varphi(x)\,dx$. *Let* α *be different from* 0 *and* $\alpha\beta$ *be greater than* 0. *Put*

$$\psi_n=\left(\frac{n\beta}{\alpha}\right)^{\frac{h}{2}}\alpha^{-n}\,\underbrace{\varphi*_{\lambda,h}\varphi*_{\lambda,h}\cdots*_{\lambda,h}\varphi}_{n}\left(\sqrt{\frac{n\beta}{\alpha}}x\right).$$

Then ψ_n *converges to* $e^{-\frac{2\pi}{\lambda}x^2}$ *as* n *tends to infinity.*

Proof. First we shall calculate the McLaurin expansion of $T_{\lambda,h}\varphi(x)$. By Proposition 5 we have

$$\frac{d}{dx}T_{\lambda,h}\varphi(0)=0,\quad \frac{d^2}{dx^2}T_{\lambda,h}\varphi(0)=-\frac{4\pi}{\lambda}T_{\lambda,h+2}\varphi(0),\quad \frac{d^3}{dx^3}T_{\lambda,h}\varphi(0)=0$$

and

$$\frac{d^4}{dx^4}T_{\lambda,h}\varphi(x)=3\left(\frac{4\pi}{\lambda}\right)^2T_{\lambda,h+4}\varphi(x)-6\left(\frac{4\pi}{\lambda}\right)^3x^3T_{\lambda,h+6}\varphi(x)+\left(\frac{4\pi}{\lambda}\right)^4x^4T_{\lambda,h+8}\varphi(x).$$

Therefore

$$\begin{aligned}T_{\lambda,h}\varphi(x)&=T_{\lambda,h}\varphi(0)-\frac{1}{2!}\frac{4\pi}{\lambda}T_{\lambda,h+2}\varphi(0)x^2+R(\theta x)x^4\\&=\alpha-\frac{2\pi}{\lambda}\beta x^2+R(\theta x)x^4,\end{aligned}$$

where $R(x)$ is a locally uniformly bounded function. Then we get

$$\begin{aligned}T_{\lambda,h}\psi_n(x)&=\alpha^{-n}T_{\lambda,h}(\varphi*_{\lambda,h}\cdots*_{\lambda,h}\varphi)\left(\frac{x}{\sqrt{\frac{n\beta}{\alpha}}}\right)\\&=\left(\alpha^{-1}T_{\lambda,h}\varphi\left(\frac{x}{\sqrt{\frac{n\beta}{\alpha}}}\right)\right)^n\\&=\left(1-\frac{2\pi}{\lambda}\cdot\frac{x^2}{n}+O\left(\frac{x^4}{n^2}\right)\right)^n\end{aligned}$$

and the last term tends to $e^{-\frac{2\pi}{\lambda}x^2}$. Because $T_{\lambda,h}$ is continuous on $\mathfrak{P}_0$, $\psi_n=T_{\lambda,h}(T_{\lambda,h}\psi_n)$ converges to $T_{\lambda,h}e^{-\frac{2\pi}{\lambda}x^2}=e^{-\frac{2\pi}{\lambda}x^2}$.

References

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