79. A Characteristic Property of L_ρ-Spaces ($\rho\geqq 1$). III

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The aim of this paper is to give a characterization of the abstract L_ρ-space[1] ($\rho\geqq 1$) in terms of the norm.

Through this paper, let $\boldsymbol{R}$ be a Banach lattice with a continuous semi-order.[2]

$\boldsymbol{R}$ is called the abstract L_ρ-space, if the norm satisfies the following condition:

(L_ρ) $\quad \|x+y\|^\rho=\|x\|^\rho+\|y\|^\rho$ for every $|x|\frown|y|=0,\ x, y\in\boldsymbol{R}$.

When we consider the case which the norm has the restricted Gateaux's differential i.e.,

(RG) $$G(x;[p]x)=\lim_{\varepsilon\to 0}\frac{\|x+\varepsilon[p]x\|-\|x\|}{\varepsilon}$$

exists for each $\|x\|\leqq 1$ and each projector $[p]$,[3] it is easily seen that for numbers α,β and projectors $[p],[q]$

(1) $$G(x;\alpha[p]x+\beta[p]y)=\alpha G(x;[p]x)+\beta G(x;[q]x)$$

if the right side has a sense.

Used the condition (RG), our characterization is described in the following form.

Theorem. *Suppose that $\boldsymbol{R}$ is at least three dimensional space. In order that $\boldsymbol{R}$ is the abstract L_ρ-space for some $\rho\geqq 1$, it is necessary and sufficient that the norm on $\boldsymbol{R}$ satisfies the conditions* (RG) *and*

(*) $$G(a+x;a)=G(a+y;a)$$

for every $a\frown x=a\frown y=0$ and $\|a+x\|=\|a+y\|=1$.

Remark. It is known that the Gateaux's differential produces the equality in the Hölder's inequality. In this sense, our theorem is closely related to the previous paper [4 and 5], especially, if the conjugately similar transformation $\boldsymbol{T}$ preserves the norm then $\|a+x\|=\|a+y\|=1$ and $a\frown x=a\frown y=0$ imply

$$G(a+x;a)=\frac{(a,\boldsymbol{T}(a+x))}{\|\boldsymbol{T}(a+x)\|}=\frac{(a,\boldsymbol{T}a)}{\|\boldsymbol{T}(a+x)\|}=\frac{(a,\boldsymbol{T}(a+y))}{\|\boldsymbol{T}(a+y)\|}=G(a+y;a)$$

because for $\|x\|=1$ we have $(x,\boldsymbol{T}x)=\|\boldsymbol{T}x\|$ and hence $G(x;[p]x)$

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1) See [3: p. 312]. The braquet [·] denotes the number of the reference in the last.

2) A semi-order is said to be *continuous*, if for any $x_\nu\downarrow_{\nu=1}^\infty$ and $0\leqq x_\nu\in\boldsymbol{R}$ there exists x such that $x_\nu\downarrow_{\nu=1}^\infty x$.

3) For any $p\in\boldsymbol{R}$, $[p]x=\bigcup_{n=1}^\infty(|p|\frown nx^+)-\bigcup_{n=1}^\infty(|p|\frown nx^-)$ where $x^+=x\smile 0$ and $x^-=(-x)^+$.

$=([p]x, \boldsymbol{T}x/\|\boldsymbol{T}x\|)$.[4] Therefore, Theorem includes the result in the paper [5].

To prove this theorem, we shall study the indicatrix of $\boldsymbol{R}$. In two-dimensional Euclidean space, the curve C is called the *indicatrix*[5] if it satisfies the following conditions:

1) C is symmetric in respect to the axices,
2) C passes through the four points $(1, 0), (0, 1), (1, 0)$ and $(0, 1)$,
3) C is the convex continuous curve.

Particularly, the curve $C(a, b)$:

$$\{(\xi, \eta);\ \|\xi a+\eta b\|=1\} \quad \text{for } a\frown b=0 \text{ and } \|a\|=\|b\|=1,$$

is called the *indicatrix of* $\boldsymbol{R}$.

Lemma 1.[6] *Suppose that* $\boldsymbol{R}$ *has at least three elements* a, b, c *which are mutually orthogonal and* $\|a\|=\|b\|=\|c\|=1$. *If* $\boldsymbol{R}$ *has only one indicatrix of* $\boldsymbol{R}$, *then either the indicatrix* C *of* $\boldsymbol{R}$ *is*

$$|\xi|^{\rho}+|\eta|^{\rho}=1 \qquad \textit{for some } \rho\geqq 1$$

or

$$\operatorname{Max}\{|\xi|, |\eta|\}=1.$$

Lemma 2. *When* $\boldsymbol{R}$ *satisfies the condition* (RG), *the function* $\eta=\eta(\xi)$ *which is defined by the indicatrix*:

$$\|\xi a+\eta b\|=1 \qquad (a\frown b=0,\ \|a\|=\|b\|=1 \textit{ and } \xi, \eta\geqq 0)$$

is differentiable and non-increasing in $0\leqq\xi<1$. (*Here, the derivative at* $\xi=0$ *means the right derivative at* $\xi=0$.)

Proof. The function $\eta=\eta(\xi)$ which is defined by the indicatrix in $0\leqq\xi, \eta$, is a one-valued concave continuous function in $0\leqq\xi<1$. Since the concave function has one-side derivatives $D^{\pm}\eta(\xi)$, putting $D^{+}\eta(\xi_0)=A$ for a fixed point $0<\xi_0<1$, we have for any $\varepsilon>0(\xi_0+\varepsilon<1)$

(2)
$$\eta(\xi_0+\varepsilon)=\eta_0+\varepsilon(A+h(\varepsilon)),\ (\eta_0=\eta(\xi_0)),$$
$$\lim_{\varepsilon\to+0} h(\varepsilon)=0$$

and hence

$$0=\|(\xi_0+\varepsilon)a+\eta(\xi_0+\varepsilon)b\|-1=\|(\xi_0a+\eta_0b)+\varepsilon(a+Ab+h(\varepsilon)b)\|-1.$$

By the triangle inequality on the norm, we have

$$\begin{aligned} 0&=\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\{\|(\xi_0a+\eta_0b)+\varepsilon(a+Ab+h(\varepsilon)b)\|-1\} \\ &\leqq \lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\{\|(\xi_0a+\eta_0b)+\varepsilon(a+Ab)\|-1\} \\ &=G(\xi_0a+\eta_0b; a+Ab) \quad \text{(by the condition (RG))}. \end{aligned}$$

On the other hand, we have

$$G(\xi_0a+\eta_0b; a+Ab)\leqq 0$$

because

$$\|(\xi_0a+\eta_0b)+\varepsilon(a+Ab)\|-\varepsilon\cdot|h(\varepsilon)|\leqq\|(\xi_0a+\eta_0b)+\varepsilon(a+Ab+h(\varepsilon)b)\|=1.$$

Therefore, we have

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4) For example, see [2, p. 114].

5) See [4, p. 342].

6) See [4, Satz II. 6].

(3) $$G(\xi_0 a+\eta_0 b;\, a+Ab)=0.$$

Similarly, putting $D^{-}\eta(\xi_0)=B$ we have

(4) $$G(\xi_0 a+\eta_0 b;\, a+Bb)=0.$$

On account of (1), (3), and (4), we have $A=B$ and moreover, by (3), (4), and the relation: $G(\xi_0 a+\eta_0 b;\, \xi_0 a+\eta_0 b)=1$,

(5) $$D\eta(\xi_0)=-\frac{G(\xi_0 a+\eta_0 b;\, a)}{G(\xi_0 a+\eta_0 b;\, b)} \text{ and } G(\xi_0 a+\eta_0 b;\, b)\neq 0 \text{ for } 0<\xi_0<1.$$

Furthermore if follows that $G(\xi a+\eta b;\, a)$ and $G(\xi a+\eta b;\, b)$ are non-negative and consequently $\eta=\eta(\xi)$ is non-increasing in $0\leqq\xi<1$. Thus Lemma is proved.

The proof of Theorem. Necessity: In the abstract L_ρ-space ($\rho\geqq 1$), it is seen that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\|x+\varepsilon[p]x\|-\|x\|\}&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\|[p^{\perp}]x^{7)}+(1+\varepsilon)[p]x\|-\|x\|\}\\ &=\lim_{\varepsilon\to 0}\{\|[p^{\perp}]x\|^{\rho}+|1+\varepsilon|^{\rho}\cdot\|[p]x\|^{\rho}\}^{\frac{1-\rho}{\rho}}\cdot|1+\varepsilon|^{\rho-1}\cdot\|[p]x\|^{\rho}\\ &=\|x\|^{1-\rho}\cdot\|[p]x\|^{\rho}\end{aligned}$$

for any $x\in \boldsymbol{R}$ and projector $[p]$, and also

$$G(a+x;\, a)=\|a\|^{\rho}\cdot\|a+x\|^{1-\rho}=G(a+y;\, a)$$

for $a\frown x=a\frown y=0$ and $\|a+x\|=\|a+y\|=1$.

Sufficiency: Since $\boldsymbol{R}$ is three dimensional, we can consider the indicatrices $C(a, b)$ and $C(a, c)$ for the mutually orthogonal elements a, b, and c with $\|a\|=\|b\|=\|c\|=1$.

For any two points $(\xi,\eta)\in C(a, b)$ and $(\xi,\zeta)\in C(a, c)$ we obtain, on the assumptions,

(6) $$G(\xi a+\eta b;\, a)=G(\xi a+\zeta c;\, a).$$

Furthermore, from the relation:

$$G(\xi a+\eta b;\, \xi a+\eta b)=1=G(\xi a+\zeta c;\, \xi a+\zeta c)$$

we have $\quad \eta\cdot G(\xi a+\eta b;\, b)=\zeta\cdot G(\xi a+\zeta c;\, c)$

and consequently,

$$\frac{1}{\eta}\cdot\frac{G(\xi a+\eta b;\, a)}{G(\xi a+\eta b;\, b)}=\frac{1}{\zeta}\,\frac{G(\xi a+\zeta c;\, a)}{G(\xi a+\zeta c;\, c)}\quad(\xi\neq 1).$$

Accordingly, by (5) it follows that

$$\frac{1}{\eta}D\eta(\xi)=\frac{1}{\zeta}D\zeta(\xi)\quad(0<\xi<1)$$

and hence $\eta(\xi)=\zeta(\xi)$ $(0\leqq\xi\leqq 1)$, because $\eta(0)=\zeta(0)=1$ and the functions $\eta(\xi)$ and $\zeta(\xi)$ are continuous.

Thus, the indicatrix $C(a, b)$ coincides with the indicatrix $C(a, c)$ and it is easily seen that

$$\operatorname{Max}\{|\xi|, |\eta|\}\neq 1 \text{ for } 0<|\xi|<1 \text{ and } (\xi,\eta)\in C(a, b).$$

Therefore, by Lemma 1, $C(a, b)$ is respresented by the form:

$$|\xi|^{\rho}+|\eta|^{\rho}=1\quad(\rho\geqq 1)$$

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7) $[p^{\perp}]x=x-[p]x$ for $x\in\boldsymbol{R}$.

and hence $\|x\|^\rho/\|x+y\|^\rho+\|y\|^\rho/\|x+y\|^\rho=1$ for any $x, y\in \boldsymbol{R}$ with $|x|\frown|y|=0$, that is, $\boldsymbol{R}$ satisfies (L_ρ)-condition. The theorem is completed.

Finally, we note that Dr. Yamamuro recently gave a characterization of the abstract L_ρ space in terms of Beurling-Livingston's duality mapping.

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