77. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. VII

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On the assumption that the ordinary part $R(\lambda)$ of the function $S(\lambda)$ defined in the statement of Theorem 1 is a transcendental integral function, in the preceding paper [1] we have discussed, under some conditions, the relation between the distribution of ζ-points of $R(\lambda)$ and that of ζ-points of $S(\lambda)$ in the exterior of an appropriately large circle with center at the origin and have proved that, if each of $R(\lambda)$ and $S(\lambda)$ has its finite exceptional value for the exterior of that circle, the two exceptional values are identical under the same conditions as above. In the present paper, however, we shall discuss these problems without using those conditions from a different point of view.

Theorem 19. Let $S(\lambda), R(\lambda)$, and $\{\lambda_\nu\}$ be the same notations as those defined in the statement of Theorem 1; let σ be a given constant satisfying the condition $\sup_\nu|\lambda_\nu|<\sigma<\infty$; let

$$C=\sup_n\left\{\frac{1}{2\pi}\left|\int_0^{2\pi}S(\rho e^{it})e^{int}dt\right|\right\}\quad(<\infty)\quad(\sup_\nu|\lambda_\nu|<\rho<\sigma<\infty);$$

let us suppose that in the exterior of the circle $|\lambda|=\sigma$ there exists an infinite subsequence $\{z_n\}_{n=1,2,3,\dots}$ of mutually distinct ζ-points of $R(\lambda)$ such that

$$\left.\begin{array}{l}R(z_n)=\zeta\\ \sigma<|z_n|\leqq|z_{n+1}|\end{array}\right\}(n=1,2,3,\cdots),\ |z_n|\to\infty(n\to\infty),\ |z_{n+1}|-|z_n|\to 0\quad(n\to\infty),$$

and that, for two systems of appropriately chosen positive numbers r_n and ε_n with $\varepsilon_n<C, n=1,2,3,\cdots$, satisfying the conditions $r_n\to 0$ $(n\to\infty)$ and $\inf_n\varepsilon_n\equiv\varepsilon>0$ respectively, $|R(z_n+r_ne^{i\theta})-\zeta|$ is greater than or equal to ε_n, irrespective of the value of $\theta\in[0,2\pi]$; let L be the least positive integer of n subject to the condition

$$|z_n|>\frac{2C\rho}{\varepsilon}+r\quad(r=\max_n r_n);$$

and let Γ_p be the circle $|\lambda-z_{L+p}|=r_{L+p}$ for each value of $p=0,1,2,\cdots$. Then, in the interior of each Γ_p, $S(\lambda)$ and $R(\lambda)$ have the same number (counted according to multiplicity) of ζ-points; and moreover, if the above hypotheses on the sets $\{z_n\}$, $\{r_n\}$, and $\{\varepsilon_n\}$ are satisfied for every $\zeta(\neq\infty)$, not exceptional value of $R(\lambda)$, and if ξ is the finite exceptional value of $S(\lambda)$ for the exterior of the circle $|\lambda|=\sigma$, ξ is also the exceptional value of $R(\lambda)$.

Proof. From now on we shall denote by $\chi(\lambda)$ the sum of the two principal parts of $S(\lambda)$. Then, as can be found from the course of the proof of Theorem 16 [2] and from the above-mentioned condition $|z_{L+p}|-r>\dfrac{2C\rho}{\varepsilon}>2\rho$, we obtain

$$|\chi(z_{L+p}+r_{L+p}e^{i\theta})|=\frac{1}{2}\left|\sum_{k=1}^{\infty}(a_k+ib_k)\left(\frac{\rho}{z_{L+p}+r_{L+p}e^{i\theta}}\right)^k\right|\leqq\frac{C\rho}{|z_{L+p}+r_{L+p}e^{i\theta}|-\rho}$$

by applying the expansion [4] of $\chi(\lambda)$ in the exterior of the circle $|\lambda|=\rho$. Since, by hypotheses, $\varepsilon\leqq|R(z_{L+p}+r_{L+p}e^{i\theta})-\zeta|$, we have therefore

$$\frac{|R(z_{L+p}+r_{L+p}e^{i\theta})-\zeta|}{|\chi(z_{L+p}+r_{L+p}e^{i\theta})|}\geqq\frac{|R(z_{L+p}+r_{L+p}e^{i\theta})-\zeta|}{\dfrac{C\rho}{|z_{L+p}+r_{L+p}e^{i\theta}|-\rho}}>\frac{\varepsilon\left(\dfrac{2C\rho}{\varepsilon}-\rho\right)}{C\rho}>1;$$

that is, the inequality $|R(z_{L+p}+r_{L+p}e^{i\theta})-\zeta|>|\chi(z_{L+p}+r_{L+p}e^{i\theta})|$ holds for every $p=0,1,2\cdots$ and every $\theta\in[0,2\pi]$. Since moreover $\chi(\lambda)$, together with $R(\lambda)$, is regular inside and on the circle Γ_p because of the fact that $|z_{L+p}|>r_{L+p}+\rho>\sup_{\nu}|\lambda_\nu|$, and since $S(\lambda)=R(\lambda)+\chi(\lambda)$ in the entire complex λ-plane, the Rouché theorem permits us to conclude that $S(\lambda)$ and $R(\lambda)$ have the same number (counted according to multiplicity) of ζ-points in the interior of each Γ_p.

Lastly we turn to the proof of the latter half of the theorem. Suppose, contrary to what we wish to prove, that ξ is not the finite exceptional value of $R(\lambda)$. Then, by the hypotheses and the first supposition concerning the sets $\{r_n\}$ and $\{\varepsilon_n\}$ associated with a subset $\{z_n\}$ of ζ-points of $R(\lambda)$ for any $\zeta(\neq\infty)$ different from its exceptional value, $S(\lambda)$ would have an infinite sequence of ξ-points tending in modulus to ∞. However this is in contradiction with the second supposition that ξ is the finite exceptional value of $S(\lambda)$ for the exterior of the circle $|\lambda|=\sigma$. Consequently ξ must be the exceptional value of $R(\lambda)$.

Remark. In fact, there exist such transcendental integral functions as are subject to the hypotheses and the suppositions given in the statement of the present theorem.

Theorem 20. Let $S(\lambda)$, $R(\lambda)$, $\{\lambda_\nu\}$, σ, ρ, and C be the same notations as before; let us suppose that in the exterior of the circle $|\lambda|=\sigma$ there exists an infinite subsequence $\{z_n\}_{n=1,2,3,\dots}$ of mutually distinct ζ-points of $S(\lambda)$ such that

$$\left.\begin{array}{l}S(z_n)=\zeta\\ \sigma<|z_n|\leqq|z_{n+1}|\end{array}\right\}\ (n=1,2,3,\cdots),\ |z_n|\to\infty\ (n\to\infty),\ |z_{n+1}|-|z_n|\to0\ (n\to\infty),$$

and that, for two systems of appropriately chosen positive numbers r_n and ε_n with $\varepsilon_n < C, n=1, 2, 3, \cdots$, satisfying the conditions $r_n \to 0$ $(n \to \infty)$ and $\inf_n \varepsilon_n \equiv \varepsilon > 0$ respectively, $|S(z_n + r_n e^{i\theta}) - \zeta|$ is greater than or equal to ε_n, irrespective of the value of $\theta \in [0, 2\pi]$; let δ be a given positive constant less than ε; let M and N be the least positive integers of n satisfying the conditions $|z_n| > \dfrac{3C\rho}{\delta} + r$ $(r = \max_n r_n)$ and $\delta \leqq |R(z_n + r_n e^{i\theta}) - \zeta|$ for all $\theta \in [0, 2\pi]$ respectively; let $L = \max(M, N)$; and let Γ_p be the circle $|\lambda - z_{L+p}| = r_{L+p}$ for each value of $p = 0, 1, 2, \cdots$. Then, in the interior of each Γ_p, $R(\lambda)$ and $S(\lambda)$ have the same number (counted according to multiplicity) of ζ-points; and moreover, if the above hypotheses on the sets $\{z_n\}$, $\{r_n\}$, and $\{\varepsilon_n\}$ are satisfied for every $\zeta (\neq \infty)$, not exceptional value of $S(\lambda)$ for the exterior of the circle $|\lambda| = \sigma$, and if ξ is the finite exceptional value of $R(\lambda)$, ξ is also the exceptional value of $S(\lambda)$ for the exterior of the circle $|\lambda| = \sigma$.

Proof. Since, as can be seen immediately from the expansion of $\chi(\lambda)$, $|\chi(\lambda)| \to 0$ $(|\lambda| \to \infty)$, and since $S(z_n + r_n e^{i\theta}) - \zeta = \{R(z_n + r_n e^{i\theta}) - \zeta\} + \chi(z_n + r_n e^{i\theta})$, there exists a positive integer G such that the inequalities $\delta \leqq |R(z_n + r_n e^{i\theta}) - \zeta|$, $n = G, G+1, G+2, \cdots$, hold for a given positive number δ less than ε, in accordance with the hypothesis $\varepsilon_n \leqq |S(z_n + r_n e^{i\theta}) - \zeta|$. Hence there exists, in fact, the least positive integer N of n satisfying the condition $\delta \leqq |R(z_n + r_n e^{i\theta}) - \zeta|$ for all $\theta \in [0, 2\pi]$. Accordingly, in the same manner as that used in the course of the proof of Theorem 19, it can be verified by hypotheses that

$$\frac{|R(z_{L+p} + r_{L+p} e^{i\theta}) - \zeta|}{|\chi(z_{L+p} + r_{L+p} e^{i\theta})|} > 2, \quad p = 0, 1, 2 \cdots.$$

In consequence, the inequalities $|S(z_{L+p} + r_{L+p} e^{i\theta}) - \zeta| > |\chi(z_{L+p} + r_{L+p} e^{i\theta})|$, $p = 0, 1, 2, \cdots$, hold for all $\theta \in [0, 2\pi]$. As can be found from the rewritten Rouché theorem stated in the course of the proof of Theorem 17 [3], this result implies that in the interior of any circle Γ_p $R(\lambda)$ has ζ-points whose number (counted according to multiplicity) is equal to that of ζ-points of $S(\lambda)$ in the interior of the same circle as it, as we were to prove.

By reasoning exactly like that applied in the proof of the latter half of Theorem 19, we can easily show the validity of the latter half of the present theorem.

References

[1] S. Inoue: Some applications of the functional-representations of normal operators in Hilbert spaces. VI, Proc. Japan Acad., **39**, 109-113 (1963).

[2] S. Inoue: Ibid., 110.
[3] ——: Ibid., 111-112.
[4] ——: Some applications of the functional-representations of normal operators in Hilbert spaces. III, Proc. Japan Acad., **38**, 641–642 (1962).

Corrections to Sakuji Inoue: "Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. VI" (Proc. Japan Acad., **39**, 109–113 (1963)).

Page 110, line 20: For "$\dfrac{C_\rho}{|z_n|-r-\rho}$" read "$\dfrac{C\rho}{|z_n|-r-\rho}$".

Page 111, line 6: For "$S(\lambda)=\zeta$" read "$S(z_n)=\zeta$".

Correction to Sakuji Inoue: "On the Functional-Representations of Normal Operators in Hilbert Spaces" (Proc. Japan Acad., **38**, 18–22 (1962)).

Page 22, line 24: For "N" read "$f(N)$".