103. *Open Mappings and Metrization Theorems*

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1963)

Let X be a T_1-space and let $\mathfrak{U}=\bigcup_{n=1}^{\infty}\mathfrak{U}_n$ be an open base of X where each $\mathfrak{U}_n$ is a point-finite system of open sets, then $\mathfrak{U}$ is called to be a σ-point-finite open base of X.

In this note, we shall obtain the necessary and sufficient condition that X has a σ-point-finite open base which is a generalization of K. Nagami's theorem [7]. As its application, we shall next obtain some metrization theorems.

1. Open images. K. Nagami [7] has shown the following theorem: *a metric space is always an open compact image*[1] *of a 0-dimensional metric space.* As a generalization of this theorem, we get the following

Theorem 1. *A T_1-space X has a σ-point-finite open base if and only if X is an open compact image of a 0-dimensional metric space.*

Proof. As the "if" part is easily seen from our previous note ([4], Theorem 5), we shall prove the "only if" part.

The following proof is carried out in the similar way as K. Nagami [7]. We may assume that X has a σ-point-finite open base $\mathfrak{U}=\bigcup_{n=1}^{\infty}\mathfrak{U}_n$ such that each $\mathfrak{U}_n=\{U_{\alpha_n}\mid\alpha_n\in A_n\}$ is a point-finite open covering of X and $\mathfrak{U}_{n+1}$ is a refinement of $\mathfrak{U}_n$ for $n=1,2,\cdots$. Let A be the set of points $a=(\alpha_n;\, n=1,2,\cdots)$ of the product space $\prod_{n=1}^{\infty}A_n$, where each A_n is a discrete topological space, such that $\bigcap_{n=1}^{\infty}U_{\alpha_n}=x$ for any point x of X. Then A is a 0-dimensional metric space as the subspace of $\prod_{n=1}^{\infty}A_n$. Let $f(a)=x$, then f is an open continuous mapping of A onto X such that $f^{-1}(x)$ is compact for any point x of X (cf. [7]). This completes the proof.

As an immediate consequence of Theorem 1 and a theorem in our previous note ([4], Theorem 5), we get the following

Theorem 2. *A T_1-space X has a σ-point-finite open base if and only if there exists a countable family $\{\mathfrak{U}_n\}$ of point-finite open coverings of X such that $\{(S(x,\mathfrak{U}_n)\mid n=1,2,\cdots\}$ is a neighborhood basis of x for each point x of X.*

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1) Let $f(X)=Y$ be an open continuous mapping. If $f^{-1}(y)$ is compact for each point y of Y, then Y is said to be an open compact image of X.

In the same way as the proof of Theorem 1, we can prove the following theorem which is a generalization of K. Nagami's theorem ([7], Theorem 3).

Theorem 3. *A T_1-space X is perfectly separable if and only if X is an open compact image of a 0-dimensional separable metric space.*

2. Metrization theorems. Theorem 4. *A collectionwise normal T_1-space X is metrizable if and only if X has a σ-point-finite open base.*

Proof. As the "only if" part is obvious, we shall prove the "if" part. By Theorem 1, X is an open compact image of a 0-dimensional metric space. Then, by the theorem in our previous note ([4], Theorem 7), X is metrizable. Thus the theorem is proved.

Remark 1. P. Alexandroff [1] has shown the following theorem: *a collectionwise normal T_1-space X is metrizable if and only if X has a uniform base.*[2)]

By Theorem 1 and a theorem due to A. Arhangelskii ([2], Theorem 1), we can see that a T_1-space X has a σ-point-finite open base if and only if X has a uniform base. Therefore Theorem 4 is equivalent to the above theorem due to P. Alexandroff.

A. H. Stone [11] has investigated the metrizability of unions of spaces. In the following, we shall obtain some theorems which are analogous to the results of A. H. Stone.

Theorem 5. *If X is a collectionwise normal space and $X=\bigcup_{n=1}^{\infty} G_n$ where each G_n is an open metrizable subset, then X is metrizable.*

Proof. It is evident that X is a T_1-space. Since G_n is an open metrizable subset of X, there exists a σ-point-finite open base $\mathfrak{U}_n$. Then, it is easy to see that $\mathfrak{U}=\bigcup_{n=1}^{\infty}\mathfrak{U}_n$ is a σ-point-finite open base of X. By Theorem 4, X is metrizable. This completes the proof.

Theorem 6. *If X is a collectionwise normal space and X is the union of a star-countable system of open metrizable subsets of X, then X is metrizable.*

Proof. Let $X=\bigcup_{\alpha\in A} G_\alpha$ where $\{G_\alpha \mid \alpha\in A\}$ is a star-countable system of open metrizable subsets. Then, $X=\bigcup_{\lambda\in A} H_\lambda$ such that $H_\lambda \cap H_{\lambda'} \neq \phi$ $(\lambda\neq\lambda')$ and each H_λ is the union of countable number of sets of $\{G_\alpha\}$ ([6], [10]). By Theorem 5, we can see that each H_λ is an open and closed metrizable subset of X. Therefore, X is metrizable ([8], [9]). This completes the proof.

2) If $\mathfrak{V}=\{V_\alpha\}$ is an open base of X such that, for every point x of X, any infinite family of sets V_α of $\mathfrak{V}$ which contain x is a neighborhood basis of x, then $\mathfrak{V}$ is called to be a uniform base of X (cf. [1]).

Theorem 7. *If X is a collectionwise normal space and X is the union of a σ-locally finite system $\mathfrak{G}=\{G_\alpha\}$ of open metrizable subsets such that the boundary $\mathfrak{B}(G_\alpha)$ of each G_α has the Lindelöf property, then X is metrizable.*

Proof. Since $\mathfrak{G}=\{G_\alpha\}$ is a σ-locally finite system, $\mathfrak{G}=\bigcup_{n=1}^{\infty}\mathfrak{G}_n$ where each $\mathfrak{G}_n=\{G_\alpha^{(n)}\}$ is a locally finite system. Since the boundary $\mathfrak{B}(G_\alpha^{(n)})$ has the Lindelöf property, there exists a countable family $\{G_{\alpha_i}|i=1,2,\cdots\}$ of sets of $\mathfrak{G}$ such that $\mathfrak{B}(G_\alpha^{(n)})\subset\bigcup_{i=1}^{\infty}G_{\alpha_i}$. Therefore $G_\alpha^{(n)}\cup(\bigcup_{i=1}^{\infty}G_{\alpha_i})\supset\overline{G_\alpha^{(n)}}$. Then we can see that $\overline{G_\alpha^{(n)}}$ has a σ-point-finite open base and $\overline{G_\alpha^{(n)}}$ is a collectionwise normal T_1-space. By Theorem 4, $\overline{G_\alpha^{(n)}}$ is a closed metrizable subset of X. Since $\mathfrak{G}_n$ is locally finite, $\{\overline{G_\alpha^{(n)}}\}$ is locally finite. Therefore $K_n=\bigcup\{\overline{G_\alpha^{(n)}}|G_\alpha^{(n)}\in\mathfrak{G}_n\}$ is a closed metrizable subset of X ([8], [9]). On the other hand, since $H_n=\bigcup\{G_\alpha^{(n)}|G_\alpha^{(n)}\in\mathfrak{G}_n\}\subset K_n$, H_n is an open metrizable subset of X and $X=\bigcup_{n=1}^{\infty}H_n$, then X is metrizable by Theorem 4. This completes the proof.

In Theorems 5 and 6, the assumption that X is a collectionwise normal space can be replaced by that X is a countably paracompact normal space. Namely, we get the following theorems.

Theorem 8. *If X is a countably paracompact normal space and $X=\bigcup_{n=1}^{\infty}G_n$ where each G_n is an open metrizable subset of X, then X is metrizable.*

Proof. Since X is countably paracompact normal space, $\{G_n\}$ has a countable, locally finite, closed refinement $\{F_n\}$ such that $F_n\subset G_n$ ([5], Proof of Theorem 3). Then X is the union of a locally finite system of closed metrizable subsets. Therefore X is metrizable ([8], [9]).

Theorem 9. *If X is a countably paracompact normal space and X is the union of a star-countable system $\{G_\alpha\}$ of open metrizable subsets of X, then X is metrizable.*

As we can easily prove Theorem 9 by the similar argument as the proof of Theorem 6, we omit the proof.

Theorem 10. *If X is a normal space and $X=\bigcup_{n=1}^{\infty}G_n$ where each G_n is an open metrizable subset of X and $\{G_n\}$ is a point-finite system, then X is metrizable.*

Proof. Since X is normal and $\{G_n\}$ is point-finite, we can easily see that there exists an open F_σ-set H_n such that $H_n\subset G_n$ for each n and $\{H_n\}$ is an open covering of X. Then, by the theorem due to H. H. Corson and E. Michael ([3], Theorem 1.1), X is metrizable. This completes the proof.

Remark 2. In Theorem 10, the hypothesis of the normality of

X can not be omitted. We can see this by the example given by H. H. Corson and E. Michael ([3], Example 6.7).

In the next place, we shall consider the case when X is the union of a family of closed metrizable subsets of X.

Theorem 11. *If X is a topological space and $X=\bigcup_{n=1}^{\infty}K_n$ where each K_n is a closed metrizable subset of X such that $\bigcap_{n=1}^{\infty}\overline{X-K_n}=\phi$, then X is metrizable.*

Proof. Let $C_n=K_n-\bigcup_{i=1}^{n-1}\mathrm{Int}(K_i)$ where $\mathrm{Int}(K_i)$ denotes the interior of K_i, then $\{C_n\}$ is a locally finite closed covering of X and each C_n is metrizable. In fact, since $\bigcap_{n=1}^{\infty}\overline{X-K_n}=\phi$, $\bigcup_{n=1}^{\infty}\mathrm{Int}(K_n)=X$. Let x be any point of X, then there exists $\mathrm{Int}(K_k)$ which contains x. Then $C_n\cap\mathrm{Int}(K_k)=\phi$ for $n>k$. Therefore $\{C_n\}$ is locally finite. It is obvious that $\{C_n\}$ is a closed covering of X. Hence X is metrizable ([8], [9]).

Remark 3. In Theorem 11, the hypothesis that $\bigcap_{n=1}^{\infty}\overline{X-K_n}=\phi$ is not superfluous even when X is collectionwise normal. We can see this by the example given by A. H. Stone ([11], p. 363).

Theorem 12. *If X is a topological space and $X=\bigcup_{\alpha\in A}K_\alpha$ where $\{K_\alpha\}$ is a σ-locally finite system of closed metrizable subsets of X such that $\bigcap_{\alpha\in A}\overline{X-K_\alpha}=\phi$, then X is metrizable.*

Proof. Since $\{K_\alpha\}$ is a σ-locally finite system, we get $\{K_\alpha\}=\bigcup_{n=1}^{\infty}\{K_\alpha^{(n)}\}$ where each $\{K_\alpha^{(n)}\}$ is locally finite. Let $Y_n=\bigcup_{\alpha\in A_n}K_\alpha^{(n)}$ where $A_n=\{\alpha\,|\,K_\alpha^{(n)}\}$, then Y_n is closed and metrizable because $\{K_\alpha^{(n)}\}$ is a locally finite system of closed metrizable subsets. Since it is evident that $\bigcap_{n=1}^{\infty}\overline{X-Y_n}=\phi$, X is metrizable by Theorem 11. This completes the proof.

Theorem 13. *If X is a paracompact topological space and X is the union of a locally countable system $\{K_\alpha\,|\,\alpha\in A\}$ of closed metrizable subsets such that $\bigcap_{\alpha\in A}\overline{X-K_\alpha}=\phi$, then X is metrizable.*

Proof. Let x be any point of X. Since $\{K_\alpha\}$ is a locally countable system, there exists an open neighborhood $U(x)$ which intersects at most a countable number of sets of $\{K_\alpha\}$. Then $\mathfrak{U}=\{U(x)\,|\,x\in X\}$ is an open covering of X. Let $\{K_i^{(x)}\,|\,i=1,2,\cdots\}=\{K_\alpha\,|\,U(x)\cap K_\alpha\neq\phi\}$ and let $\mathfrak{X}^{(n)}=\{K_n^{(x)}\,|\,x\in X\}$, then $X=\bigcup_{x=1}^{\infty}\{\bigcup_{x\in X}K_n^{(x)}\}$. We shall next prove that each $\mathfrak{X}^{(n)}$ is locally finite. Since X is paracompact, $\mathfrak{U}$ has a locally finite open refinement $\mathfrak{B}=\{V_\beta\,|\,\beta\in B\}$. For every point x of X, there exist $V_\beta\in\mathfrak{B}$ and $U(x')\in\mathfrak{U}$ such that $x\in V_\beta\subset U(x')$. Then $\{K_\alpha\,|\,K_\alpha\cap V_\beta\neq\phi\}\subset\{K_\alpha\,|\,K_\alpha\cap U(x')\neq\phi\}=\{K_i^{(x')}\,|\,i=1,2,\cdots\}$. On the other hand, since there exists an open neighborhood $W(x)$ of x which intersects only a finite number of sets of $\mathfrak{B}$, $\{K_n^{(x')}\,|\,W(x)\cap K_n^{(x')}\neq\phi,\ K_n^{(x')}\in\mathfrak{X}^{(n)},\ x'\in X\}$ is finite. Therefore $\mathfrak{X}^{(n)}$ is a locally finite system of closed metrizable subsets. Then, by Theorem 12, X is metrizable. This completes the proof.

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