101. Nörlund Summability of a Sequence of Fourier Coefficients

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1963)

1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots p_n.$$

The sequence to sequence transformation, viz.

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu} s_\nu}{P_n} = \sum_{\nu=0}^{n} \frac{p_\nu s_{n-\nu}}{P_n}, \quad (P_n \neq 0), \tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of constants $\{p_n\}$. The series $\sum a_n$ or the sequence $\{s_n\}$ is said to be summable by Nörlund means, or summable (N, p_n) to the sum s, if $\lim_{n\to\infty} t_n$ exists and equals s.

The condition of regularity of the method of summability (N, p_n) defined by (1.1) are

$$\lim_{n\to\infty} p_n/P_n = 0, \tag{1.2}$$

and

$$\sum_{k=0}^{n} |p_k| = O(P_n), \quad \text{as} \quad n \to \infty. \tag{1.3}$$

If $\{p_n\}$ is real and non-negative, (1.3) is automatically satisfied and then (1.2) is the necessary and sufficient condition for the regularity of the method of summation (N, p_n).

In the special case in which $p_n = 1/(n+1)$, and, therefore

$$P_n \sim \log n, \quad \text{as} \quad n \to \infty,$$

t_n reduces to the familiar 'harmonic mean' [4] of s_n, and if it be denoted by t'_n, then $\sum a_n$ or the sequence $\{s_n\}$ is said to be summable by harmonic means, or summable (H), to the sum s if $\lim_{n\to\infty} t'_n = s$.

If the method of summability (N, p_n) be superimposed on the Cesàro means of order one, another method of summability $(N, p_n)\cdot C_1$, is obtained [1].

2. Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x), \tag{2.1}$$

and its conjugate series is

(2.2) $$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).$$

We write

$$\psi(t) = f(x+t) - f(x-t) - l,$$
$$\Psi(t) = \int_0^t |\psi(u)|\, du,$$
$$p(1/t) = p_\tau,$$

and $P(1/t) = P_\tau$, where τ is the integral part of $1/t$.

In 1959 Varshney [6] proved the following theorem.

THEOREM. If

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left[\frac{t}{\log 1/t}\right],$$

as $t \to +0$, then the sequence $\{nB_n(x)\}$ is summable $(N, 1/(n+1))\cdot C_1$ to the value l/π.

In this paper we prove two Theorems. In the first of these we show that even if this particular sequence $\{1/(n+1)\}$ be replaced by a more general sequence $\{p_n\}$, the result will continue to hold true. In the second we give a more general condition for the $(N, p_n)\cdot C_1$ summability of the sequence $\{nB_n(x)\}$. In what follows $\{p_n\}$ is real, non-negative and non-increasing sequence such that $P_n \to \infty$ with n. We prove the following Theorems.

3. THEOREM 1. *If (N, p_n) be a regular Nörlund method, defined by a real, non-negative, monotonic non-increasing sequence of constants $\{p_n\}$, such that $P_n \to \infty$, and*

(3.1) $$\sum_{k=a}^{n} P_k/k \log k = O(P_n),$$

as $n \to \infty$, where a is a fixed positive integer; then, if

(3.2) $$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left[\frac{t}{\log 1/t}\right],$$

as $t \to +0$, the sequence $\{nB_n(x)\}$ is summable $(N, p_n)\cdot C_1$, to the value l/π.

THEOREM 2. *If (N, p_n) be a regular Nörlund method defined by a real non-negative and non-increasing sequence such that $P_n \to \infty$ with n and if*

(3.3) $$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left[\frac{p(1/t)}{P(1/t)}\right],$$

as $t \to +0$, then the sequence $\{nB_n(x)\}$ is summable $(N, p_n)\cdot C_1$ to the value l/π.

4. We require the following lemmas to prove our Theorems.

Lemma 1. [2]. (i). *For $0 < t \le \pi$, and for any n, a and b,*

$$\left|\sum_a^b p_k\, e^{i(n-k)t}\right| < AP(1/t),$$

where A is an absolute constant, and

(ii) $$\frac{1}{t}p(1/t) \leq P(1/t).$$

Lemma 2. *For* $0 \leq t \leq 1/n$,

$$|Q_n(t)| \equiv \left| \frac{1}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left(\frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right) \right| = O(n).$$

Proof.

$$\begin{aligned} |Q_n(t)| &= O\left[\frac{1}{P_n} \sum_{k=1}^{n} p_{n-k}(k^2 t) \right] \\ &= O\left[\frac{n}{P_n} \sum_{k=1}^{n} p_{n-k} \right] \\ &= O(n). \end{aligned}$$

Lemma 3. *For* $0 < t \leq \pi$,

$$|Q_n(t)| = O\left[\frac{P(1/t)}{tP_n} \right].$$

Proof. By lemma 1(i) and Abel's transformation, we have

$$\begin{aligned} |Q_n(t)| &= O\left[\frac{1}{P_n} \left| \sum_{k=1}^{n} p_{n-k} \frac{\sin kt}{kt^2} \right| \right] + O\left[\frac{P(1/t)}{tP_n} \right] \\ &= O\left[\frac{1}{tP_n} \left| \sum_{k=0}^{\tau} p_k \frac{\sin (n-k)t}{(n-k)t} \right| \right] + O\left[\frac{1}{tP_n} \left| \sum_{k=\tau+1}^{n-1} p_k \frac{\sin (n-k)t}{(n-k)t} \right| \right] \\ &\qquad + O\left[\frac{P(1/t)}{tP_n} \right] \\ &= O\left[\frac{1}{tP_n} \sum_{k=0}^{\tau} p_k \right] + O\left[\frac{1}{tP_n} \left\{ \frac{1}{t} \sum_{k=\tau+1}^{n-2} |\Delta p_k| \right\} \right] \\ &\qquad + O\left[\frac{p_{n-1}}{t^2 P_n} \right] + O\left[\frac{p_{\tau+1}}{t^2 P_n} \right] + O\left[\frac{P(1/t)}{tP_n} \right] \\ &= O\left[\frac{P(1/t)}{tP_n} \right], \end{aligned}$$

by Lemma 1(ii) and since $|\sum \sin kt/k| \leq \frac{1}{2}\pi + 1$ [5].

4. Proof of Theorem 1. Let $\sigma_n(x)$ be the $(C, 1)$ transform of the sequence $\{nB_n(x)\}$, then after Mohanty and Nanda [3], we have

$$\begin{aligned} \sigma_n(x) - l/\pi &= \frac{1}{n} \sum_{r=1}^{n} rB_r(x) - l/\pi \\ &= \frac{1}{n} \int_0^{\pi} \psi(t) \left(\frac{\sin nt}{4n \sin^2 \frac{1}{2}t} - \frac{\cos nt}{2 \tan \frac{1}{2}t} \right) \cdot dt \\ &\qquad + \frac{1}{2\pi} \int_0^{\pi} \psi(t) \sin nt \, dt + o(1) \end{aligned}$$

(4.1) $$= \frac{1}{\pi} \int_0^{\pi} \psi(t) \left[\frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] dt + o(1),$$

by Riemann-Lebesgue theorem.

On account of the regularity of the method of summability, we have to show that under our assumptions

(4.2) $$\int_0^\pi \psi(t)\frac{1}{\pi P_n}\sum_{k=1}^{n} p_{n-k}\left[\frac{\sin kt}{kt^2}-\frac{\cos kt}{t}\right]dt=o(1),$$

as $n\to\infty$.

We set

$$\begin{aligned} I&=\int_0^\pi \psi(t)\frac{1}{\pi P_n}\sum_{k=1}^{n} p_{n-k}\left[\frac{\sin kt}{kt^2}-\frac{\cos kt}{t}\right]dt\\ &=\int_0^\pi \psi(t)\,Q_n(t)\,dt\\ &=\left(\int_0^{1/n}+\int_{1/n}^{\delta}+\int_\delta^\pi\right)\psi(t)\,Q_n(t)\,dt\\ &=I_1+I_2+I_3. \end{aligned} \tag{4.3}$$

Now, by Lemma 2,

$$\begin{aligned} I_1&=O\left[\int_0^{1/n}|\psi(t)|\,|Q_n(t)|\,dt\right]\\ &=O[n\Psi(1/n)]\\ &=o\left[\frac{n}{n\log n}\right]\\ &=o(1),\ \text{as } n\to\infty. \end{aligned} \tag{4.4}$$

Again, by Lemma 3,

$$\begin{aligned} I_2&=O\left[\int_{1/n}^{\delta}|\psi(t)|\frac{P(1/t)}{tP_n}dt\right]\\ &=O\left[\frac{1}{P_n}\left\{\Psi(t)\frac{P(1/t)}{t}\right\}_{1/n}^{\delta}\right]\\ &\qquad+O\left[\frac{1}{P_n}\int_{1/n}^{\delta}\frac{\Psi(t)\,P(1/t)}{t^2}dt\right]+o(1)\\ &=o(1)+o\left[\frac{1}{P_n}\left\{\frac{P(1/t)}{\log 1/t}\right\}_{1/n}^{\delta}\right]+o\left[\frac{1}{P_n}\int_{1/n}^{\delta}\frac{P(1/t)}{t\log 1/t}dt\right]\\ &=o(1)+o\left[\frac{1}{P_n}\int_{\frac{1}{\delta}+1}^{n}\frac{P(x)}{x\log x}dx\right],\\ &\qquad\qquad\text{where } x=1/t,\\ &=o(1)+o\left[\frac{1}{P_n}\sum_{[1/\delta]+1}^{n}\frac{P_k}{k\log k}\right]\\ &=o(1). \end{aligned} \tag{4.5}$$

Since the method of summation is regular, we have

(4.6) $$I_3=o(1),$$

as $n\to\infty$, by Riemann-Lebesgue theorem.

This completes the proof of the Theorem 1.

PROOF OF THEOREM 2. Here also we have to show that, under the condition (3.3),

$$I=o(1).$$

By Lemma 2 and hypothesis (3.3)

$$I_1=O[n\Psi(1/n)]$$
$$=o[np_n/P_n]$$
$$=o(1), \tag{4.7}$$

since $np_n \leq P_n$.

Again, by Lemma 3,

$$I_2=O\left[\int_{1/n}^{\delta}|\psi(t)|\frac{P(1/t)}{tP_n}dt\right]$$
$$=O\left[\frac{1}{P_n}\left\{\Psi(t)\frac{P(1/t)}{t}\right\}_{1/n}^{\delta}\right]$$
$$+O\left[\frac{1}{P_n}\int_{1/n}^{\delta}\frac{\Psi(t)P(1/t)}{t^2}dt\right]+o(1)$$
$$=o(1)+o\left[\frac{1}{P_n}\left\{\frac{p(1/t)}{t}\right\}_{1/n}^{\delta}\right]+o\left[\frac{1}{P_n}\int_{1/n}^{\delta}\frac{p(1/t)}{t^2}dt\right]$$
$$=o(1)+o\left[\frac{1}{P_n}\int_{1/\delta}^{n}p(x)\,dx\right]$$
$$=o(1). \tag{4.8}$$

Since the method of summation is regular, we have

$$I_3=o(1), \tag{4.9}$$

as $n\to\infty$, by Riemann-Lebesgue theorem.

This proves the Theorem 2.

I am very much indebted to Professor B. N. Prasad for his kind interest and advice in the preparation of this paper.

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