100. *Isometries on Hilbert Spaces*

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1. The structure of an operator A on an infinite dimensional Hilbert space is closely related to the type of the smallest von Neumann algebra containing A (which is called a von Neumann algebra generated by A and denoted $\mathscr{R}(A)$ in what follows).*) That is to say, the von Neumann algebras offer a powerful tool for studying general operators on Hilbert spaces. An operator A on a Hilbert space is said to be of type I (II, III) if a von Neumann algebra $\mathscr{R}(A)$ is of type I (II, III). In particular, an operator A is said to be of type I_∞ if $\mathscr{R}(A)$ is a factor of type I_∞. A normal operator A is the simplest case of type I where $\mathscr{R}(A)$ is abelian. In the general case, it is very desirable to discriminate the type I, II and III among known operators. The situation of this paper is the first step in that direction and we shall prove the following

THEOREM. *An isometry on a Hilbert space is of type I.*

The proof will proceed by stages. For the terminology of von Neumann algebras, we shall always refer to [2].

2. Let $\boldsymbol{K}$ be a Hilbert space and let $\boldsymbol{H}$ the space of sequences

$$\varphi=\{\varphi_0, \varphi_1, \varphi_2, \cdots\}$$

of vectors in $\boldsymbol{K}$ such that $\sum_{n=0}^{\infty}\|\varphi_n\|^2<\infty$. If an operator V is defined on $\boldsymbol{H}$ by

$$V\{\varphi_0, \varphi_1, \varphi_2, \cdots\}=\{0, \varphi_0, \varphi_1, \varphi_2, \cdots\},$$

then it is clear that V is a non-unitary isometry on $\boldsymbol{H}$ and is called the unilateral shift on $\boldsymbol{H}$. In the case $\dim K=\mathfrak{N}$, V may also be called the unilateral shift of multiplicity $\mathfrak{N}$. If the unilateral shift has multiplicity one, it is said to be simple. M. H. Stone [4] has shown that the simple unilateral shift is irreducible (i.e., it has no non-trivial reducing subspaces). That is to say, a von Neumann algebra generated by the simple unilateral shift on $\boldsymbol{H}$ is the algebra $\mathscr{L}(H)$ of all operators on $\boldsymbol{H}$. This implies the following

LEMMA 1. *The simple unilateral shift is of type I_∞.*

We shall generalize the above lemma to the general unilateral shift. Actually, with the aid of the tensor product of von Neumann algebras, we shall prove the following

LEMMA 2. *The unilateral shift V on the Hilbert space $\boldsymbol{H}$ is*

*) By an operator, we always understand a bounded linear transformation.

type I_∞.

PROOF. Let $\{\varepsilon_n\}$ $(n=0,1,2,\cdots)$ be the orthonormal base in $l_2(N)$ where N is the set of non-negative integers. Each vector of the tensor product $\boldsymbol{K}\otimes l_2(N)$ of $\boldsymbol{K}$ and $l_2(N)$ is uniquely expressed in the following form

$$\sum_{n=0}^{\infty}\varphi_n\otimes\varepsilon_n,$$

where $\{\varphi_n\}$ is a sequence of vectors in $\boldsymbol{K}$ such that $\sum_{n=0}^{\infty}\|\varphi_n\|^2<\infty$. Defines a linear transformation W of $\boldsymbol{H}$ to $\boldsymbol{K}\otimes l_2(N)$ by

$$W\{\varphi_0,\varphi_1,\cdots,\varphi_n,\cdots\}=\sum_{n=0}^{\infty}\varphi_n\otimes\varepsilon_n,$$

then W is an isometry of $\boldsymbol{H}$ onto $\boldsymbol{K}\otimes l_2(N)$ and the straightforward computation shows that

$$WVW^{-1}\Big(\sum_{n=0}^{\infty}\varphi_n\otimes\varepsilon_n\Big)=\sum_{n=0}^{\infty}\varphi_n\otimes\varepsilon_{n+1}.$$

This implies that $WVW^{-1}=I_{\boldsymbol{K}}\otimes v$, where $I_{\boldsymbol{K}}$ is the identity operator on $\boldsymbol{K}$ and v is the simple unilateral shift on $l_2(N)$. Note that $\mathcal{R}(I_{\boldsymbol{K}}\otimes v)$ is the ampliation of $\mathcal{R}(v)$ by [2; Ch. 1, § 2, Prop. 6], then it follows from Lemma 1 that $\mathcal{R}(I_{\boldsymbol{K}}\otimes v)$ is of type I_∞. Since a von Neumann algebra $\boldsymbol{M}$ generated by V is clearly spatial isomorphic to $\mathcal{R}(I_{\boldsymbol{K}}\otimes v)$, we obtain that $\boldsymbol{M}$ is of type I_∞. This means that V is of type I_∞.

Here we mention that the description of all reducing subspaces of the unilateral shift due to P. R. Halmos [3; Theorem 1] is directly deduced from the proof of the above lemma.

COROLLARY. *An operator A on $\boldsymbol{H}$ commutes with both V and V^* if and only if $A=\hat{c}$ for some operator c on $\boldsymbol{K}$, where $\hat{c}$ is an inflated operator on $\boldsymbol{H}$ defined by* $\hat{c}\{\varphi_0,\varphi_1,\varphi_2,\cdots\}=\{c\varphi_0,c\varphi_1,c\varphi_2,\cdots\}$.

In fact, by the isometry W, $\boldsymbol{M}$ (resp. $\boldsymbol{M}'$) is spatial isomorphic to the ampliation $C_{\boldsymbol{K}}\otimes\mathcal{R}(v)$ (resp. $\mathcal{L}(K)\otimes C_{l_2(N)}$) of $\mathcal{R}(v)$ (resp. $\mathcal{L}(K)$). Thus an operator A belongs to $\boldsymbol{M}'$ if and only if WAW^{-1} belongs to $\mathcal{L}(K)\otimes C_{l_2(N)}$. Consequently, since W carries an inflated operator $\hat{c}$ on $\boldsymbol{H}$ to an operator $c\otimes I_{l_2(N)}$, we obtain the desired result.

3. Now we concentrate our attention to the isometries on a Hilbert space and the structure theorems of the isometries which have been shown by A. Brown [1] and P. R. Halmos [3] are provided. Let U be an isometry on a Hilbert space $\boldsymbol{H}$ and let $\boldsymbol{K}$ the orthogonal complement of the range of U in what follows. The result obtained in [3; Lemma 1] is stated as follows.

LEMMA 3. *The subspaces $\{U^n\boldsymbol{K}\}$ $(n=0,1,2,\cdots)$ are mutually orthogonal and $\big(\sum_{n=0}^{\infty}U^n\boldsymbol{K}\big)^\perp=\bigcap_{n=0}^{\infty}U^n\boldsymbol{H}$. That is,*

$$\boldsymbol{H}=\Big(\sum_{n=0}^{\infty}U^n\boldsymbol{K}\Big)\oplus\Big(\bigcap_{n=0}^{\infty}U^n\boldsymbol{H}\Big).$$

The preceding lemma will illustrate the fact that an isometry is precisely decomposed as the direct sum of a unitary operator and an unilateral shift as seen below (cf. [1; Lemma 2.1]).

LEMMA 4. *Let $\boldsymbol{M}$ be a von Neumann algebra generated by U, then there exists a projection P in the center of $\boldsymbol{M}$ such that the restriction U_P of U to $P\boldsymbol{H}$ is a unitary operator and U_{I-P} is unitary equivalent to the unilateral shift on $\boldsymbol{K}\otimes l_2(N)$.*

PROOF. Let P be a projection on $\bigcap_{n=0}^{\infty} U^n \boldsymbol{H}$, then, by Lemma 3, $I-P$ is a projection on $\sum_{n=0}^{\infty} U^n \boldsymbol{K}$. Since the subspaces $\bigcap_{n=0}^{\infty} U^n \boldsymbol{H}$ and $\sum_{n=0}^{\infty} U^n \boldsymbol{K}$ are invariant by U, the subspace $P\boldsymbol{H}=\bigcap_{n=0}^{\infty} U^n \boldsymbol{H}$ reduces U, and so $PU=UP$. Thus $P\in\boldsymbol{M}'$ where $\boldsymbol{M}'$ is the commutant of $\boldsymbol{M}$. On the other hand, for any unitary element U' of $\boldsymbol{M}'$, $U'[U^n\boldsymbol{H}]=U^n\boldsymbol{H}$ and $U'^*[U^n\boldsymbol{H}]=U^n\boldsymbol{H}$, which yield that the subspace $P\boldsymbol{H}=\bigcap_{n=0}^{\infty} U^n \boldsymbol{H}$ reduces U'. Thus $PU'=U'P$ for all unitary element U' of $\boldsymbol{M}'$, and so $P\in\boldsymbol{M}$. Consequently, P belongs to the center $\boldsymbol{M}\cap\boldsymbol{M}'$ of $\boldsymbol{M}$.

Next, assume that $\varphi\in P\boldsymbol{H}$, $\varphi=U\psi$ for some $\psi\in\boldsymbol{H}$. Then, for $\xi\in\boldsymbol{K}$,

$$\langle U^n\xi, \psi\rangle=\langle U^{n+1}\xi, U\psi\rangle=\langle U^{n+1}\xi, \varphi\rangle=0,$$

which yields $\psi\in(\sum_{n=0}^{\infty} U^n\boldsymbol{K})^{\perp}=P\boldsymbol{H}$. This means that U_P is unitary. Defines a linear transformation W of $(I-P)\boldsymbol{H}$ to $\boldsymbol{K}\otimes l_2(N)$ by

$$W(\sum_{n=0}^{\infty} U^n\varphi_n)=\sum_{n=0}^{\infty}\varphi_n\otimes\varepsilon_n,$$

then W is an isometry from $(I-P)\boldsymbol{H}$ onto $\boldsymbol{K}\otimes l_2(N)$, and

$$WU_{I-P}W^{-1}(\sum_{n=0}^{\infty}\varphi_n\otimes\varepsilon_n)=W(\sum_{n=0}^{\infty}U^{n+1}\varphi_n)=\sum_{n=0}^{\infty}\varphi_n\otimes\varepsilon_{n+1}$$

shows that $WU_{I-P}W^{-1}$ is the unilateral shift on $\boldsymbol{K}\otimes l_2(N)$.

PROOF OF THEOREM. With the notation established above, it follows from Lemma 2 that a von Neumann algebra $\mathcal{R}(WU_{I-P}W^{-1})$ is of type I_∞. This implies that $\mathcal{R}(U_{I-P})$ is of type I_∞ since $\mathcal{R}(U_{I-P})$ is spatial isomorphic to $\mathcal{R}(WU_{I-P}W^{-1})$. Clearly, the operator U_P is of type I. Keeping in mind that P is a projection in the center of $\boldsymbol{M}$, it is easy to see that the von Neumann algebra $\mathcal{R}(U_P)$(resp. $\mathcal{R}(U_{I-P})$ is the restriction $\boldsymbol{M}_{P\boldsymbol{H}}$(resp. $\boldsymbol{M}_{(I-P)\boldsymbol{H}}$) of $\boldsymbol{M}$ to $P\boldsymbol{H}$(resp. $(I-P)\boldsymbol{H}$). That is,

$$\boldsymbol{M}=\boldsymbol{M}_{P\boldsymbol{H}}\oplus\boldsymbol{M}_{(I-P)\boldsymbol{H}}=\mathcal{R}(U_P)\oplus\mathcal{R}(U_{I-P}).$$

Making use [2; Ch. I, §8, Prop. 1], we conclude that $\boldsymbol{M}$ is of type I, which completes the proof.

References

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[2] J. Dixmier: Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris (1957).
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