93. *Note on Balayage and Maximum Principles*

By Masanori KISHI
Mathematical Institute, Nagoya University

(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1963)

1. Let Ω be a locally compact Hausdorff space, every compact subset of which is separable, and G be a positive lower semicontinuous kernel on Ω such that $G(x, y)$ is locally bounded at any point $(x, y)\in\Omega\times\Omega$ with $x\neq y$. The *adjoint* kernel $\check{G}$ of G is defined by $\check{G}(x, y)=G(y, x)$. Given a positive measure μ, its potential $G\mu(x)$ and adjoint potential $\check{G}\mu(x)$ are defined by

$$G\mu(x)=\int G(x, y)d\mu(y) \quad \text{and} \quad \check{G}\mu(x)=\int \check{G}(x, y)d\mu(y)$$

respectively.

This note is a summary of some relations among the balayage principle and related maximum principles in the potential theory. The details will be published later elsewhere.

2. **Definitions.** We say that a property holds *G-p.p.p.* on a subset $X\subset\Omega$, when any compact subset of the set of points in X at which the property is missing does not support any positive measure $\nu\neq 0$ with finite G-energy $\int G\nu d\nu$.

(I) *Continuity principle.* For any positive measure μ with compact support $S\mu$, if the restriction of $G\mu(x)$ to $S\mu$ is finite and continuous, then $G\mu(x)$ is finite and continuous in the whole space Ω.

(II) *Balayage principle.* For any compact set K and any positive measure μ, there exists a positive measure μ', supported by K, such that

$$G\mu'(x)\leq G\mu(x) \quad \text{in } \Omega,$$
$$G\mu'(x)=G\mu(x) \quad G\text{-}p.p.p. \text{ on } K.$$

(III) *Equilibrium principle.* For any compact set K, there exists a positive measure μ, supported by K, such that

$$G\mu(x)\leq 1 \quad \text{in } \Omega,$$
$$G\mu(x)=1 \quad G\text{-}p.p.p. \text{ on } K.$$

(IV) *Domination principle.* For a positive measure μ with compact support and finite G-energy and for a positive measure ν with compact support, an inequality $G\mu(x)\leq G\nu(x)$ on $S\mu$, the support of μ, implies the same inequality in Ω.

(V) *Maximum principle.* For a positive measure μ with compact support, the validity of an inequality $G\mu(x)\leq 1$ on $S\mu$ implies that of the same inequality in Ω.

(VI) *Complete maximum principle.* For a positive measure μ with compact support and finite G-energy and for a positive measure ν with compact support, if an inequality $G\mu(x)\leq G\nu(x)+a$ with a non-negative number a holds on $S\mu$, then the same inequality holds in Ω.

(VII) *Strong maximum principle.* If μ is a positive measure with compact support and finite G-energy and ν is a positive measure with compact support such that $G\mu(x)\leq G\nu(x)+a$ on $S\mu\cup S\nu$ with $a\geq 0$, then the same inequality holds in Ω.

3. Theorems. The following is fundamental.

Theorem 1. *Assume that the adjoint kernel $\check{G}$ satisfies the continuity principle. If $u(x)$ is a positive finite upper semicontinuous function on a compact set K, then there exists a positive measure μ, supported by K, such that*

$$G\mu(x)\geq u(x) \quad G\text{-p.p.p. on } K,$$
$$G\mu(x)\leq u(x) \quad \text{everywhere on } S\mu.$$

This follows from

Theorem 2. *Given positive finite numbers a_{ki} and u_k $(k, i=1, 2, \cdots, n)$, there exist non-negative finite numbers t_i $(i=1, 2, \cdots, n)$ such that*

$$\sum_{i=1}^{n} a_{ki}t_i\geq u_k \quad \text{for } k=1,2,\cdots,n,$$
$$\sum_{i=1}^{n} a_{ji}t_i=u_j \quad \text{for every } j \text{ with } t_j\neq 0.$$

Theorem 3. *If both G and its adjoint $\check{G}$ satisfy the continuity principle, then the following four statements are equivalent;*

(1) *G satisfies the balayage principle,*

(2) *$\check{G}$ satisfies the balayage principle,*

(3) *G satisfies the domination principle,*

(4) *$\check{G}$ satisfies the domination principle.*

Contrary to these equivalences, even if G satisfies the equilibrium principle, the adjoint $\check{G}$ does not in general, except for specific kernels, for example convolution kernels. We can state only

Theorem 4. *When G and $\check{G}$ satisfy the continuity principle, G satisfies the equilibrium principle if and only if G satisfies the maximum principle.*

Theorem 5. *Let $\check{G}$ satisfy the continuity principle. Then G satisfies the complete maximum principle if and only if G satisfies the domination and maximum principles.*

Theorem 6. *Let $\check{G}$ satisfy the continuity principle. Then G satisfies the complete maximum principle if and only if G does the strong maximum principle.*

The complete maximum principle can be also characterized by the following *complete balayage principle.*

Theorem 7. *Let G and $\check{G}$ satisfy the continuity principle. Then G satisfies the complete maximum principle if and only if G satisfies the complete balayage principle, that is, for any positive measure μ with compact support and any compact set K there exists a positive measure μ', supported by K, such that*

$$G\mu'(x)=G\mu(x)+1 \quad G\text{-p.p.p. on } K$$
$$G\mu'(x)\leq G\mu(x)+1 \quad \text{in } \Omega.$$

Now let N be another positive lower semicontinuous kernel.

(VIII) *Balayage principle with respect to N.* For any compact set K and any positive measure μ with compact support, there exists a positive measure μ', supported by K, such that

$$G\mu'(x)=N\mu(x) \quad G\text{-p.p.p. on } K$$
$$G\mu'(x)\leq N\mu(x) \quad \text{in } \Omega.$$

(IX) *Domination principle with respect to N.* For any positive measure μ with compact support and finite G-energy and for any positive measure ν with compact support, the inequality $G\mu(x)\leq N\nu(x)$ on $S\mu$ implies the same inequality in Ω.

Theorem 8. *Let G and $\check{G}$ satisfy the continuity principle. Then G satisfies the domination principle with respect to N, if and only if G satisfies the balayage principle with respect to N.*

4. Comments. For symmetric kernels G(i.e. $G\equiv\check{G}$) Theorem 1 is well-known. It is verified by using the Gauss-Ninomiya variation. For non-symmetric kernels the variation is useless in its original form.

Theorem 2 follows from Kronecker's existence theorem [1] in the theory of combinatorial topology.

Ninomiya [6] first obtained Theorem 3 for symmetric kernels. Deny [4] followed to show the equivalence between (1) and (4) for strictly increasing diffusion kernels. Choquet and Deny [3] obtained Theorem 3 for regular kernels on a compact space which consists of a finite number of points.

Theorem 4 was obtained by Ninomiya [6] for symmetric kernels.

The complete maximum principle was first introduced by Cartan and Deny [2]. They obtained Theorem 5 for symmetric kernels of positive type.

Theorems 6 and 7 are new. The former is an answer to the question raised by Deny [5].

The balayage and domination principles with respect to N were introduced by Ninomiya [7]. Theorem 8 is a generalization of the result of Ninomiya who discussed symmetric kernels.

Theorems 6 and 7 in [6] can be verified for our non-symmetric kernels.

References

[1] P. Alexandroff and H. Hopf: Topologie I, Berlin (1935).

[2] H. Cartan and J. Deny: Le principe du maximum en théorie du potentiel et la notion de fonction surharmonique, Acta Sci. Math. Szeged, **12**, 81–100 (1950).

[3] G. Choquet and J. Deny: Modèles finis en théorie du potentiel, Jour. Anal. Math., **5**, 77–135 (1956–57).

[4] J. Deny: Les deux aspects de la théorie du potentiel, Sém. Bourbaki, No. 148, 18 pp. (1957).

[5] ——: Les principes du maximum en théorie du potentiel, Sém. Brelot-Choquet-Deny, 6e année, No. 10, 8 pp. (1962).

[6] N. Ninomiya: Étude sur la théorie du potentiel pris par rapport au noyau symétrique, Jour. Inst. Polytech. Osaka City Univ., **6**, 147–179 (1957).

[7] ——: Sur le problème du balayage generalisé, Jour. Math. Osaka City Univ., **12**, 115–138 (1961).