92. A Property of Certain Differentiable Manifolds

By Seiya SASAO

Department of Mathematics, Musashi Institute of Technology, Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., Sept. 12, 1963)

Let M be a compact oriented differentiable manifold without boundary which satisfies the following conditions:

(1) M is $(n-1)$-connected,

(2) $\dim M=2n+1$, $n\equiv 0 \pmod 2$.

Then the oriented cobordism class of M is determined by a Stiefel-Whitney number $W_n\cdot W_{n+1}[M]$, for other Stiefel-Whitney numbers and all Pontryagin numbers vanish. In this paper we shall show that the property of M to be cobordant to zero can be represented by a property of $H_n(M, Z)$. In case $n=2$ this was done by Wall [1].

We shall prove the following

Theorem. $W_n\cdot W_{n+1}[M]=0 \Longleftrightarrow H_n(M, Z)\approx F\oplus T\oplus T$

$W_n\cdot W_{n+1}[M]\neq 0 \Longleftrightarrow H_n(M, Z)\approx F\oplus T\oplus T\oplus Z_2$

where F and T denote a free abelian group and a torsion group respectively and Z_2 is the group of order 2, $\oplus$ denotes the direct sum.

The proof will be given in several steps.

$H_n(M, Z)$ can be decomposed as follows:

$$H_n(M, Z)=\sum_{i=1}^{a_0} Z[\bar{u}_i]+\sum_p\sum_i\sum_{j=1}^{a_i(p)} Z_{p^i}[\bar{u}_p^{i,j}],$$

where p runs over all prime numbers, $\bar{u}_i$, $\bar{u}_p^{i,j}$ denote generators. Since we are interested in $a_i(p)$, it is sufficient for us to consider $H^n(M, Z_p)$ and $H^{n+1}(M, Z_p)$, i.e.

$$H^n(M, Z_p)=\sum_{i=1}^{a_0} Z_p[u_i]+\sum_{i=1}\sum_{j=1}^{a_i(p)} Z_{p^i}[u_p^{i,j}]$$

$$H^{n+1}(M, Z_p)=\sum_{i=0}^{a_0} Z_p[v_i]+\sum_{i=1}\sum_{j=1}^{a_i(p)} Z_{p^i}[v_p^{i,j}].$$

Now we consider a matrix $A=(a_{s,t})$ over Z_p defined by $a_{s,t}=u_p^{j,l}\cdot v_p^{i,k}[M]$ for $a_0+\sum_{m=1}^{j-1} a_m(p)<s\leqq a_0+\sum_{m=1}^{i} a_m(p)$, $a_0+\sum_{m=1}^{i-1} a_m(p)<t \leqq a_0+\sum_{m=1}^{i} a_m(p)$ where j, l, i, k are given by $s=a_0+\sum_{m=1}^{j-1} a_m(p)+l$, $t=a_0+\sum_{m=1}^{i-1} a_m(p)+k$, $i, j\geqq 1$, and $a_{s,t}=u^s\cdot v^t[M]$ for $1\leqq s, t\leqq a_0$.

By Poincaré duality we have $\det A\neq 0$. Let Δ_p^i denote the higher Bockstein operator. As we can take $v_p^{i,k}=\Delta_p^i(u_p^{i,k})$ $(i\geqq 1)$, we obtain

Lemma 1. If p is odd, we have

(1) $u^k\cdot v_p^{i,j}=0$

(2) $u_p^{i,j}\cdot v_p^{s,t}=0$ $(s<i)$

(3) $u_p^{i,j}\cdot v_p^{i,t}=-u_p^{i,t}\cdot v_p^{i,j}$ $(j\neq t)$

(4) $u_p^{i,j}\cdot v_p^{i,j}=0$;

thus A has the following form

$$A=\left|\begin{array}{c|c|c|c}
A_0 & \multicolumn{3}{c}{0} \\ \hline
 & A_1 & \multicolumn{2}{c}{*} \\ \cline{2-4}
* & 0 & A_2 & * \\ \cline{3-4}
 & & 0 &
\end{array}\right.$$

where $A_0, A_1, A_2\cdots$ are antisymmetric and have zero diagonal.

Since $\det A=\det A_0\cdot\det A_1\cdot\det A_2\cdots$ we have $\det A_i\neq 0$. Hence the degree of A_i must be even.

Lemma 2. If p is 2, we have

(1) $u_2^k v_2^{i,j}=0$

(2) $u_2^{i,j}v_2^{s,t}=0$ $(i>s)$

(3) $u_2^{i,j}v_2^{i,t}=0$ $(j\neq t)$

(4) $u_2^{i,j}v_2^{i,j}=0$ $(i>1)$

$u_2^{1,j}v_2^{1,j}=S_q^n v_2^{1,j}$.

In this case $A_0, A_2, A_3,\cdots$ are antisymmetric and have zero diagonal, but A_1 has not always zero diagonal. However, if $n\neq 2, 4, 8$, S_q^n is known to be decomposable, so that $u_2^{1,j}\cdot v_2^{1,j}=S_q^n v_2^{1,j}=0$, A_1 has zero diagonal. Since A_1 is antisymmetric degree of A_1 is also even, therefore $H_n(M, Z)\cong F\oplus T\oplus T$. On the other hand, in these case $n\neq 2, 4, 8$ the decomposability of S_q^n and Wu's formula between Stiefel-Whitney classes and squaring operations imply that M is cobordant to zero. Now consider the case $n=2, 4, 8$. A computation of matrix using Wu's formula shows $W_n\cdot W_{n+1}[M]=a_1(2)$. Then it is easy to see the conclusion of our theorem.

Reference

[1] C. T. C. Wall: Killing the middle homotopy groups of odd dimensional manifolds, Trans. Amer. Math. Soc., **103**, 421-433 (1962).