91. Boundary Convergence of Blaschke Products in the Unit-Circle

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(1) **Introduction.** Let $B(z)$ be Blaschke products:

$$B(z)=\prod_{n=1}^{\infty} b(z, a_n),$$

where

$$b(z, a)=\bar{a}/|a| \cdot (a-z)/(1-\bar{a}z),$$
$$0<|a_n|<1 \quad (n=1, 2, \cdots),$$
$$\sum_{n=1}^{\infty}(1-|a_n|)<+\infty. \tag{1.1}$$

In this note, we shall establish the following two theorems on boundary convergence of Blaschke-products.

Theorem 1 is concerned with the necessary and sufficient condition for $B(z)$ to be regular at $z=e^{i\theta}$:

Theorem 1. *If $z=e^{i\theta}$ is not the limiting point of $\{a_n\}$, then $B(z)$ is absolutely and uniformly convergent to a regular function in the neighborhood of $z=e^{i\theta}$.*

As its immediate consequences, we get

Corollary 1. *For $B(z)$ to be singular at $z=e^{i\theta}$, it is necessary and sufficient that $z=e^{i\theta}$ is the limiting point of $\{a_n\}$.*

Corollary 2. *If $B(z)$ is regular at $z=e^{i\theta}$, then $B(z)$ is uniformly and absolutely convergent in the neighborhood of $z=e^{i\theta}$.*

In the preceding paper ([2] 4–5), the author proved Corollary 1 by somewhat complicated method.

Theorem 2 is of Abelian type:

Theorem 2. *If $B(z)$ is absolutely convergent at $z=e^{i\theta}$, then $B(z)$ tends uniformly to $B(e^{i\theta})$ as $z\to e^{i\theta}$ within Stolz-domain with vertex at $z=e^{i\theta}$.*

As its consequence, we have

Corollary 3. *If $B(z)$ is absolutely convergent at $z=e^{i\theta}$, then $B(re^{i\theta})$ is continuously defined for $0\leqq r\leqq +\infty$ by the unique formula: $\prod_{n=1}^{+\infty} b(re^{i\theta}, a_n)$.*

Corollaries 2 and 3 are remarkable phenomena, whose analogy in the case of Taylor series cannot exist evidently.

(2) **Proof of Theorem 1.** By the simple computation,

$$B(z)=\prod_{n=1}^{+\infty}\{1+c(z, a_n)\}, \tag{2.1}$$

where

$$c(z, a)=(1-|a|)/|a|-(1-|a|^2)/|a|(1-\bar{a}z).$$

Because $z=e^{i\theta}$ is not the limiting point of $\{a_n\}$, we can find two positive constants ε and $\delta(\varepsilon)$ such that

$$|z-1/\bar{a}_n|\geqq\varepsilon \text{ for } |z-e^{i\theta}|\leqq\delta(\varepsilon).$$

Hence, by (2.1)

$$|c(z, a_n)|\leqq(1-|a_n|)/|a_n|+(1-|a_n|^2)/|a_n|^2\varepsilon \text{ for } |z-e^{i\theta}|\leqq\delta(\varepsilon),$$

so that, by (1.1) $\sum_{n=1}^{+\infty}c(z, a_n)$ is absolutely and uniformly convergent in $|z-e^{i\theta}|\leqq\delta(\varepsilon)$. Since $c(z, a_n)$ is regular in $|z-e^{i\theta}|\leqq\delta(\varepsilon)$, $B(z)=\prod_{n=1}^{+\infty}\{1+c(z, a_n)\}$ is absolutely and uniformly convergent to the regular function in $|z-e^{i\theta}|\leqq\delta(\varepsilon)$, which is to be proved.

(3) **Proof of Theorem 2.** By (2.1)

$$(1-|a_n|)/|e^{i\theta}-a_n|\cdot(1+1/|a_n|)\leqq|c(e^{i\theta}, a_n)|+(1-|a_n|)/|a_n|,$$

so that, by (1.1) and $\sum_{n=1}^{+\infty}|c(e^{i\theta}, a_n)|<+\infty$, it follows that

$$\sum_{n=1}^{+\infty}(1-|a_n|)/|e^{i\theta}-a_n|<+\infty. \tag{3.1}$$

By the inequality:

$$|1-\bar{a}_nre^{i\theta}|>r|a_n-e^{i\theta}| \text{ for } 0<r<1,$$

$$|c(re^{i\theta}, a_n)|\leqq(1-|a_n|)/|a_n|+(1-|a_n|)/|e^{i\theta}-a_n|\cdot(1+1/\alpha)\cdot1/\beta \tag{3.2}$$

for $0<\alpha\leqq|a_n|, 0<\beta\leqq r<1$. Taking account of (1.1), (3.1) and (3.2), $\sum_{n=1}^{+\infty}c(re^{i\theta}, a_n)$ is absolutely and uniformly convergent for $0<\beta\leqq r<1$. Hence, $B(re^{i\theta})=\prod_{n=1}^{+\infty}b(re^{i\theta}, a_n)$ is uniformly convergent for $0<\beta\leqq r<1$. Since $\lim_{r\to1}\prod_{n=1}^{N}b(re^{i\theta}, a_n)=\prod_{n=1}^{N}b(e^{i\theta}, a_n)$, by the uniform convergence of $B(re^{i\theta})$ ([1] p. 339) we have

$$\lim_{r\to1}\lim_{N\to+\infty}\prod_{n=1}^{N}b(re^{i\theta}, a_n)=\lim_{N\to+\infty}\lim_{r\to1}\prod_{n=1}^{N}b(re^{i\theta}, a_n),$$

so that

$$\lim_{r\to1}B(re^{i\theta})=B(e^{i\theta}).$$

Therefore, by the boundedness of $B(z)$ in $|z|<1$, and E. Lindelöf's theorem

$$\lim_{\substack{z\to e^{i\theta}\\ z\in S}}B(z)=B(e^{i\theta}),$$

where S is Stolz domain with vertex at $z=e^{i\theta}$.

(4) **Proof of Corollary 3.** For $|z|>1$, we can put

$$1/B(z)=\prod_{n=1}^{+\infty}b(1/z, \bar{a}_n).$$

By the convergence of $\sum_{n=1}^{+\infty}(1-|\bar{a}_n|)$, $1/B(z)$ is Blaschke products defined in $|z|>1$. If $B(e^{i\theta})$ is absolutely convergent, then $1/B(e^{i\theta})$ is also absolutely convergent because of $1/B(e^{i\theta})=\prod_{n=1}^{+\infty}(1+c_n)^{-1}=\prod_{n=1}^{+\infty}(1-c_n+O(c_n^2))$, where $c_n = c(e^{i\theta}, a_n)$.

Hence, by Theorem 2

$$\lim_{r\to 1+0} 1/B(re^{i\theta}) = 1/B(e^{i\theta})$$

so that, again by Theorem 2

$$\lim_{r\to \pm 10} B(re^{i\theta}) = B(e^{i\theta}).$$

Therefore, $B(re^{i\theta})$ is continuously defined for $0 \leqq r \leqq +\infty$ by the unique formula: $\prod_{n=1}^{+\infty} b(re^{i\theta}, a_n)$, provided that $\prod_{n=1}^{+\infty} b(e^{i\theta}, a_n)$ is absolutely convergent.

References

[1] K. Knopp: Theory and Application of Infinite Series, London and Glasgow (1928).
[2] C. Tanaka: On functions of class U, Ann. Acad. Sci. Fenn. I.A., 1–12 (1962).