130. On an Example of Non-uniqueness of Solutions of the Cauchy Problem for the Wave Equation

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1. **Introduction.** In the recent note [4] F. John has constructed the following example: For any positive integer m there exists a solution of the wave equation $\Box u=(\partial^2/\partial x^2+\partial^2/\partial y^2-\partial^2/\partial t^2)u=0$, which is analytic in a cyrindrical domain $\mathcal{D}=\{(x, y, t); x^2+y^2<1\}$ and belongs to C^m in R^3 not C^{m+2} in the neighborhood of any point outside $\mathcal{D}$.

The purpose of this note is to construct real valued functions u, f and g which belong to $\mathcal{B}$ and satisfy the equation $Lu\equiv(\Box+f\,\partial/\partial t+g)u=0$ in R^3, where the support of u equals to the set $R^3-\mathcal{D}$. What is remarkable is that the cylinder $S=\{(x, y, t); x^2+y^2=1\}$ is non-characteristic for L. Hence this example shows that for the operator L the uniqueness of solutions of the Cauchy problem for the non-characteristic surface S does not hold. But we must remark that any solution for the equation with the principal part $\Box$, which has its support in a 'strictly convex set' at a point of a time-like plane, vanishes identically in a neighborhood of that point (see [5]).

Many examples of non-uniqueness have been constructed by A. Pliś [6] and [7], P. Cohen [1] etc., and L. Hörmander has proved in the general theory that the uniqueness for an operator with the principal part $\Box$ does not hold even for a time-like plane if we admit complex valued coefficients (see [3] p. 228). But our example is interesting in the physical meaning and we can take $f=0$ if we admit complex valued g and u.

We shall construct this by the method of A. Pliś [7], using the asymptotic expansion of Bessel functions $J_\lambda(\lambda a)$ in the interval $(0, 1-\lambda^{-2\rho/5}]$ for a fixed ρ $(0<\rho<1)$.

2. **Lemma 1.** *Let $J_\lambda(a)$ be Bessel functions of order $\lambda>0$. Then, for any fixed $\rho(0<\rho<1)$ we have the following asymptotic formula:*

$$(1)\qquad J_\lambda(\lambda a)=(2\pi\lambda\tanh\alpha)^{-1/2}\exp\{\lambda(\tanh\alpha-\alpha)\}(1+0(\lambda^{-1/5}))$$
$$(0<a<1,\ \cosh\alpha=a^{-1},\ \alpha>0)$$

which is valid uniformly for every a in $(0, 1-\lambda^{-2\rho/5}]$.

Proof. First of all we remark

$$(2)\qquad 1\geqq\tanh\alpha=\sqrt{1-a^2}\geqq\lambda^{-\rho/5}\ \text{in}\ 0<a\leqq1-\lambda^{-2\rho/5}.$$

We shall use a well-known integral representation of Bessel functions (see [2] p. 412):

$$J_\lambda(\lambda a)=\frac{1}{2\pi}\int_{\Gamma_0}\exp\{\lambda(-ia\sin\zeta+i\zeta)\}d\zeta\ \ (\zeta=u+iv)$$

where Γ_0 consists of three sides of a rectangle with vertices at $-\pi+i\infty$, $-\pi$, π and $\pi+i\infty$, and is oriented from $-\pi+i\infty$ to $\pi+i\infty$.

Setting $f(\zeta)=-ia\sin\zeta+i\zeta$ we have $f(\zeta)=(a\cos u\sinh v-v)+i(u-a\sin u\cosh v)$. It is clear that we can deform Γ_0 to a curve defined by $\Gamma: u-a\sin u\cosh v=0$ without varing the values of $J_\lambda(\lambda a)$. Then we have

$$J_\lambda(\lambda a)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{\lambda g(u)\}du$$

where $g(u)$ is defined by

$$g(u)=a\cos u\sinh v-v$$

$$(\cosh v(u)=u/(a\sin u)\ (u\neq 0)\text{ and }v(0)=\alpha).$$

First we evaluate $g(u)$ in $-\lambda^{-2/5}\leqq u\leqq\lambda^{-2/5}$. Since

$$\left|\frac{dv(u)}{du}\right|=\left|\frac{1}{\sinh v}\left(\frac{u}{a\sin u}\right)\right|\leqq\frac{1}{\sqrt{u^2/(a\sin u)^2-1}}\cdot\frac{C|u|}{a}$$

$$\leqq C|u|/\sqrt{1-a^2}\leqq C\lambda^{-(2-\rho)/5}\text{ by (2),}$$

we have $|v-\alpha|\leqq C\lambda^{-(2-\rho)/5}|u|\leqq C\lambda^{-(4-\rho)/5}$ in $-\lambda^{-2/5}\leqq u\leqq\lambda^{-2/5}$. Hence by Taylor expansion

$$f(u+iv)=f(i\alpha)+\{u+i(v-\alpha)\}f'(i\alpha)+\{u+i(v-\alpha)\}^2f''(i\alpha)/2$$
$$+\frac{1}{2}\{u+i(v-\alpha)\}^3\int_0^1(1-\theta)^2f'''(i\alpha+\theta\{u+i(v-\alpha)\}d\theta,$$

we have

(3) $$g(u)=f(u+iv(u))=(\tanh\alpha-\alpha)-u^2\tanh\alpha/2+0(\lambda^{-6/5}).$$

Here we must remark $f'(i\alpha)=0$ and $|f'''(i\alpha+\theta\{u+i(v-\alpha)\})|\leqq 2$. Consequently we have

$$\int_{-\lambda^{-2/5}}^{\lambda^{-2/5}}\exp\{\lambda g(u)\}du=\exp\{\lambda(\tanh\alpha-\alpha)\}$$
$$\times\int_{-\lambda^{-2/5}}^{\lambda^{-2/5}}\exp\left\{-\frac{\lambda}{2}u^2\tanh\alpha\right\}du(1+0(\lambda^{-1/5}))$$
$$=\exp\{\lambda(\tanh\alpha-\alpha)\}\frac{1}{\sqrt{\lambda\tanh\alpha}}\left[\int_{-\infty}^{\infty}\exp\left\{-\frac{w^2}{2}\right\}dw\right.$$
$$\left.-\left\{\int_{-\infty}^{-\lambda^{1/10}\sqrt{\tanh\alpha}}+\int_{\lambda^{1/10}\sqrt{\tanh\alpha}}^{\infty}\right\}\exp\left\{-\frac{w^2}{2}\right\}dw\right](1+0(\lambda^{-1/5})).$$

Remarking $\lambda^{1/10}\sqrt{\tanh\alpha}\geqq\lambda^{1/10-\rho/10}=\lambda^{(1-\rho)/10}$ by (2), we get

$$\frac{1}{2\pi}\int_{-\lambda^{-2/5}}^{\lambda^{-2/5}}\exp\{\lambda g(u)\}du=(2\pi\lambda\tanh\alpha)^{-1/2}\exp\{\lambda(\tanh\alpha-\alpha)\}(1+0(\lambda^{-1/5})).$$

Since $g'(u)\gtrless 0$ in $0\lessgtr u\lessgtr\pm\pi$ by easy computation, we have $g(u)\leqq\text{Max}\{g(\lambda^{-2/5}),g(-\lambda^{-2/5})\}$, and by (3) and $\lambda^{-4/5}\tanh\alpha\geqq\lambda^{-(4+\rho)/2}$ we have

$$\left\{\int_{-\pi}^{-\lambda^{-2/5}}+\int_{\lambda^{-2/5}}^{\pi}\right\}\exp\{\lambda g(u)\}du=0\cdot\exp\{\lambda(\tanh\alpha-\alpha)\}\exp\{-\lambda^{(1-\rho)/5}/3\}$$
$$=0(\lambda^{-1/5})(2\pi\lambda\tanh\alpha)^{-1/2}\exp\{\lambda(\tanh\alpha-\alpha)\}.$$

Q.E.D.

Lemma 2. *Consider* $G_m(r)=J_{m^6}(m^6 r)$ *in the interval* $1-Mm^{-2}\leqq r\leqq 1-m^{-2}$ *with any fixed constant* $M>1$. *Then we have*

$$G_m(r)=(1+o(1))(2\pi^2 l)^{-1/4}m^{-5/2}\exp\{(1+o(1))2\sqrt{2}/3\cdot l^{3/2}m^3\} \quad (r=1-lm^{-2}). \tag{4}$$

Remark: It is essential in the following discussion that the exponent of l is larger than 1.

Proof. In (1) we set $\lambda=m^6$ and $\rho=5/6$, then we have

$$G_m(r)=(2\pi)^{-1/2}(1-r^2)^{-1/4}m^{-3}\left\{\frac{r\exp(\sqrt{1-r^2})}{1+\sqrt{1-r^2}}\right\}^{m^6}(1+o(1)) \quad (0<r\leqq 1-m^{-2}). \tag{5}$$

Set $f(r)=r(r+\sqrt{1-r^2})^{-1}\exp(\sqrt{1-r^2})$. Then, as $f'(r)=r^{-2}(1-\sqrt{1-r^2})\sqrt{1-r^2}\exp(\sqrt{1-r^2})=(1+o(1))\sqrt{2}\sqrt{1-r}$ in $1-Mm^{-2}\leqq r\leqq 1$, we have

$$f(r)=1-\sqrt{2}(1+o(1))\int_r^1\sqrt{1-r}\,dr=(1+o(1))2\sqrt{2}/3\cdot(1-r)^{3/2}.$$

Hence, for $r=1-lm^{-2}(1\leqq l\leqq M)$ we have by (5)

$$G_m(r)=(1+o(1))(2\pi^2 l)^{-1/4}m^{-5/2}\{1-(1+o(1))2\sqrt{2}/3\cdot l^{3/2}m^{-3}\}^{m^6}$$
$$=(1+o(1))(2\pi^2 l)^{-1/4}m^{-5/2}(e+o(1))^{-(1+o(1))2\sqrt{2}/3\cdot l^{3/2}m^3},$$

and get (4). Q.E.D.

Lemma 3. *Set* $F_m(r)=G_m(m^{-1}(m-1)r)$ *and* $r_m(s)=1+m^{-1}-sm^{-1}(m+1)^{-1}(0\leqq s\leqq 1)$. *Then* $F_m(r)$ *satisfy differential equations*

$$F''_m(r)+r^{-1}F'_m(r)-(m^6r^{-2}-m^4(m-1)^2)F_m(r)=0 \tag{6}$$

and

$$F_m(r_m(s))=(1+o(1))(2\pi^2(1+s))^{-1/4}m^{-5/2}\times\exp\{(1+o(1))2\sqrt{2}/3\cdot(1+s)^{3/2}m^3\} \quad (0\leqq s\leqq 1). \tag{7}$$

Furthermore, if we determine γ_{m+1} *such as*

$$\gamma_{m+1}F_{m+1}(r_{m+1}(2^{-1}))=F_m(r_{m+1}(2^{-1})) \quad (r_m(s)=1+m^{-1}-sm^{-1}(m+1)^{-1},\ 0\leqq s\leqq 1). \tag{8}$$

then we have

$$\gamma_{m+1}\leqq\exp\{-m^3\} \tag{9}$$

and

$$\begin{cases}\text{i)}\ \ \gamma_{m+1}F_{m+1}(r_{m+1}(s))\leqq C\exp\{-m^3/15\}F_m(r_{m+1}(s)) & (0\leqq s\leqq 1/4)\\ \text{ii)}\ \ F_m(r_{m+1}(s))\leqq C\exp\{-m^3/15\}\gamma_{m+1}F_{m+1}(r_{m+1}(s)) & (3/4\leqq s\leqq 1)\end{cases} \tag{10}$$

for sufficiently large m.

Proof. (6) is clear, and because of $m^{-1}(m-1)r_m(s)=1-(1+s+0(m^{-1}))m^{-2}$ we get (7) by (4).

Since $F_m(r_{m+1}(s))=F_m(r_m(1+s+0(m^{-1})))$, applying the mean value theorem such as $x^{3/2}-y^{3/2}=3/2\cdot\sqrt{\theta}(x-y)$ $(x\lesseqgtr\theta\lesseqgtr y)$ we get (9) by (7), and writing

$$\frac{F_m(r_{m+1}(s))}{\gamma_{m+1}F_{m+1}(r_{m+1}(s))}=\frac{F_m(r_{m+1}(s))}{F_m(r_{m+1}(2^{-1}))}\cdot\frac{F_m(r_{m+1}(2^{-1}))}{\gamma_{m+1}F_{m+1}(r_{m+1}(2^{-1}))}\cdot\frac{\gamma_{m+1}F_{m+1}(r_{m+1}(2^{-1}))}{\gamma_{m+1}F_{m+1}(r_{m+1}(s))}$$

we get (10).

3. Theorem. *There exist real valued functions* u_0, f_0 *and* g_0

of class $\mathcal{B}$ in R^3 which satisfy the equation

(11) $$\square[u_0]=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial t^2}\right)u_0=\left(f_0\frac{\partial}{\partial t}+g_0\right)u_0,$$

where supp u_0[1] *equals to $R^3-\mathcal{D}(\mathcal{D}=\{(x, y, t); x^2+y^2<1\})$.*

Proof. Set $u_m(r, \theta, t)=F_m(r)\cos(m^6\theta+m^4(m-1)^2t)$ $(r>1)$. Then, by (6) we have

$$L[u_m]\equiv(\partial^2/\partial r^2+r^{-1}\partial/\partial r+r^{-2}\partial^2/\partial\theta^2-\partial^2/\partial t^2)u_m=0.$$

Take functions $A(r)$ and $A_M(r)\in C^\infty_{(0,\infty)}$ such that

$$A(r)=\begin{cases}1 \text{ in a nbd.}^{2)} \text{ of } [1/8, 7/8]\\ 0 \text{ in a nbd. of } [1, \infty),\end{cases}$$

and for sufficiently large $M>1$ to be fixed later

(12) $$A_M(r)=\begin{cases}0 \text{ for } r\leqq 1+(M+2)^{-1}\\ 1 \text{ for } r\geqq 1+(M+2)^{-1}+1/4\cdot(M+1)^{-1}(M+2)^{-1}.\end{cases}$$

Set $A_m(r)=A(m^2/2\cdot\{r-(1+m^{-1})\}+1)$ $(m>M)$.

Then, we have for sufficiently large m

(13) $$A_m(r)=\begin{cases}1 \text{ in a nbd. of } (I_{m+1}-I_{m+1,4})\cup(I_m-I_{m,1})^{3)}\\ 0 \text{ in } (0, 1+(m+1)^{-1}]\cup(1+m^{-1}, \infty).\end{cases}$$

Now we define $u(r, \theta, t)\in C^\infty_{(r>1)}$ by

(14) $$u(r, \theta, t)=A_M(r)u_M+\sum_{m=M+1}^{\infty}\gamma_{M+1}\cdot\cdots\cdot\gamma_m A_m(r)u_m$$

with γ_m defined by (8), and set

(15) $$\begin{cases}f=g=0 \text{ in } K=[1+(M+1)^{-1}, \infty)\cup\{\bigcup_{m=M+1}^{\infty}(I_{m,2}\cup I_{m,3})\}\\ f=L[u]\dfrac{u_{|t}}{u^2+u_{|t}^2} \text{ and } g=L[u]\dfrac{u}{u^2+u_{|t}^2} \text{ in } K^c,^{4)}\end{cases}$$

where $u_{|t}=\partial/\partial t\, u$ and K^c=the complement of K in $(1, \infty)$.

By the recursion formula of Bessel functions

(16) $$2J'_\lambda=J_{\lambda-1}+J_{\lambda+1}$$

and (7), we have for $1+(m+2)^{-1}\leqq r\leqq 1+m^{-1}$

$|d^k/dr^k F_m(r)|\leqq C_k\exp\{o(m^3)\}F_m(r)$.

Hence, for $1+(m+2)^{-1}\leqq r\leqq 1+(m+1)^{-1}$ $(m\geqq M)$, we have by (7) and (16)

(17) $$|D^k u|^{5)}\leqq C_k\gamma_{M+1}\cdot\cdots\cdot\gamma_m m^{6k}(F_m+\gamma_{m+1}F_{m+1})\exp\{o(m^3)\}.$$

As $\gamma_{M+1}\cdot\cdots\cdot\gamma_m\leqq\exp\{-m^4/5+C_1M^4\}$ by (9), remarking $1+(m+2)^{-1}\leqq r\leqq 1+(m+1)^{-1}$ we get by (7).

(18) $$|D^k u|\leqq C_k\exp\{-C_2m^4\}\leqq C_k\exp\{-C_2(r-1)^{-4}/16\}\to 0 \quad (r\searrow 1).$$

1) supp u=the closure of $\{(x, y, t);\ u(x, y, t)\neq 0\}$.

2) 'nbd.' is the abbreviation of 'neighborhood'.

3) $I_m=[1+(m+1)^{-1}, 1+m^{-1}]$ and
$I_{m,k}=[1+m^{-1}-k/4\cdot m^{-1}(m+1)^{-1}, 1+m^{-1}-(k-1)/4\cdot m^{-1}(m+1)^{-1}]$ $(k=1,\cdots,4)$.

4) If we admit complex valued coefficients, taking
$u_m(r, \theta, t)=F_m(r)\exp\{\sqrt{-1}(m^6\theta+m^4(m-1)^2t)\}$, $f=0$ and $g=L[u]/u$
we can continue the similar discussion.

5) $|D^k u|=\left\{\sum_{i+j+l=k}\left|\dfrac{\partial^k}{\partial r^i\partial\theta^j\partial t^l}u\right|^2\right\}^{1/2}$.

Now, consider $u^2+u_{|t}^2$ in $I_{m+1,1}$ for $m\geqq M$. Since

$$u^2+u_{|t}^2\geqq\gamma_{M+1}^2\cdot\cdots\cdot\gamma_m^2\{(u_m^2+u_{m|t}^2)/2-\gamma_{m+1}^2A_{m+1}^2(u_{m+1}^2+u_{m+1|t}^2)\}$$
$$\geqq\gamma_{M+1}^2\cdot\cdots\cdot\gamma_m^2\{F_m(r)^2/2-C_3m^6\gamma_{m+1}^2F_{m+1}(r)^2\},$$

we have by i) of (10)

(19) $$u^2+u_{|t}^2\geqq 3^{-1}\gamma_{M+1}^2\cdot\cdots\cdot\gamma_m^2F_m(r)^2>0 \text{ in } I_{m+1,1},$$

and so by ii) of (10) we have

(20) $$u^2+u_{|t}^2\geqq 3^{-1}\gamma_{M+1}^2\cdot\cdots\cdot\gamma_m^2\gamma_{m+1}^2F_{m+1}(r)^2>0 \text{ in } I_{m+1,4}.$$

Hence, as $L[u]=0$ in a neighborhood of $I_{m+1,2}\smile I_{m+1,3}$, we have f and g are of class $C^\infty_{(r>1)}$. As $L[u]=L[\gamma_{M+1}\cdot\cdots\cdot\gamma_{m+1}A_{m+1}u_{m+1}]$ in $I_{m+1,1}$, we have by (16) and i) of (10)

(21) $$|D^kL[u]|\leqq C_{k+2}\gamma_{M+1}\cdot\cdots\cdot\gamma_{m+1}m^{6(k+2)}\exp\{o(m^3)\}F_{m+1}(r).$$

We can write

$$|D^kf|\leqq\left|D^k\left\{L[u]\frac{u_{|t}}{u^2+u_{|t}^2}\right\}\right|$$
$$=\sum_{\substack{i_1+\cdots+i_\nu=2k+1\\ \nu\leqq 2(k+1)}}a_{i_1\cdots i_\nu}\frac{|D^{i_1}L[u]|}{(u^2+u_{|t}^2)^{1/2}}\times\left\{\frac{|D^{i_2}u|}{(u^2+u_{|t}^2)^{1/2}}\cdot\cdots\cdot\frac{|D^{i_\nu}u|}{(u^2+u_{|t}^2)^{1/2}}\right\}.$$

Hence, by (17), (19), and (21) we have

$$|D^kf|\leqq C_k'm^{12(k+1)}\{\gamma_{m+1}F_{m+1}(r)/F_m(r)\}\exp\{o(m^3)\},$$

and using i) of (10) we get in $I_{m+1,1}$

(22) $$|D^kf|\leqq C_k''\exp\{-m^3/16\}.$$

By (17), (20) and ii) of (10) it is clear that we can get (22) in $I_{m+1,4}$ and further for g in $I_{m+1,1}\smile I_{m+1,4}$. Hence, for $1+(m+2)^{-1}\leqq r\leqq 1+(m+1)^{-1}$ we get by (15)

(23) $$|D^kf|,\ |D^kg|\leqq C_k\exp\{-m^3/16\}\leqq C_k\exp\{-(r-1)^3/16^2\}\to 0\quad(r\searrow 1).$$

Now, we take the non-singular transformation: $x=r\cos\theta$, $y=r\sin\theta (r>0)$. Then, L takes the form □. If we define for a sufficiently large fixed M, $u_0=f_0=g_0=0$ in $\bar{\mathscr{D}}=$ the closure of $\mathscr{D}$ and $u_0=u, f_0=f$ and $g_0=g$ defined by (14) and (15) respectively in $\mathscr{D}^c$ with $x=r\cos\theta$, $y=r\sin\theta$, then it is clear that u_0, f_0 and g_0 satisfy the desired conditions by the periodicity of u, f, g and (18), (23), and the boundedness of Bessel functions. Q.E.D.

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