# 129. A Note on the Logarithmic Means

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§ 1. When a sequence $\{s_n\}$ is given we define the logarithmic means by the transformation

$$t_0=s_0,\quad t_1=s_1,$$

(1) $$t_n=\frac{1}{\log n}\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)\quad(n\geq 2).$$

If $\{t_n\}$ tends to a finite limit $s$ as $n\to\infty$, we shall denote that $\{s_n\}$ is summable $(l)$ to $s$. (See [2] p. 59, p. 87.)

As is well known the Cesàro means $(C,1)$ are defined by the transformation

(2) $$\sigma_n=\frac{1}{n+1}(s_0+s_1+\cdots+s_n)\quad(n\geq 0).$$

Concerning these methods of summability we know the following

**Theorem 1.** *If $\{s_n\}$ is summable $(C,1)$ to $s$, then it is summable $(l)$ to the same sum. There is a sequence summable $(l)$ but not summable $(C,1)$.* (See [2] p. 59, [7] p. 32.)

We shall prove, in this note, some converse of this theorem.

**Theorem 2.** *If $\{s_n\}$ is summable $(l)$, with*

$$\frac{1}{\log n}\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)=s+o\left(\frac{1}{\log n}\right),$$

*then $\{s_n\}$ is also summable $(C,1)$. The condition $o\left(\frac{1}{\log n}\right)$ cannot be replaced by $O\left(\frac{1}{\log n}\right)$.*

*Proof.* From (1) and (2) we get

$$s_0=t_0,\quad s_1=t_1,\quad s_2=3\left(t_2\log 2-t_0-\frac{t_1}{2}\right),$$

$$s_n=(n+1)\{t_n\log n-t_{n-1}\log(n-1)\}\quad(n\geq 3),$$

and

(3) $$\begin{aligned}\sigma_n&=\frac{1}{n+1}(s_0+s_1+\cdots+s_n)\\&=\frac{-1}{n+1}\left\{2t_0+\frac{1}{2}t_1+t_2\log 2+t_3\log 3+\cdots+\right.\\&\qquad\left.+t_{n-1}\log(n-1)\right\}+t_n\log n.\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{-1}{n+1}\left\{2+\frac{1}{2}+\log 2+\log 3+\cdots+\log(n-1)\right\}+\log n$$

$$=\lim_{n\to\infty}\log\frac{n}{\sqrt[n+1]{n!}}=1,$$

we may suppose $s=0$ without loss of generality. Since $t_n\log n=o(1)$ by assumption, we can easily see, from (3), $\sigma_n=o(1)$.

To prove the second part of this theorem we put

$$t_0=t_1=0,\quad t_n=\frac{(-1)^n}{\log n}\quad (n\geq 2),$$

or

$$s_0=s_1=0,\quad s_2=3,$$
$$s_n=2(-1)^n(n+1)\quad (n\geq 3).$$

For this sequence we see

$$\lim_{n\to\infty}t_n=0\ \text{ and }\ t_n=O\Big(\frac{1}{\log n}\Big),$$

but the sequence $\{\sigma_n\}$ cannot lead to a limit, whence the proof is complete.

The first part of Theorem 2 may be easily generalized as follows:

**Corollary.** *If $\{s_n\}$ is summable $(l)$, with*

$$\lim_{n\to\infty}\Big\{\frac{1}{\log n}\Big(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\Big)-s\Big\}\log n=\alpha,$$

*where $\alpha$ is a finite value, then $\{s_n\}$ is also summable $(C,1)$.*

We can prove it from (3) quite similarly as in the case of Theorem 2.

**§ 2.** In previous papers the author established some theorems on the summability methods $(l)$ and $(L)$. Here the method $(L)$ is defined by the sequence-to-function transformation

$$\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$

for $x\to 1-0$. (See [1], [3] p. 81.)

The author proved the following theorems. (See [4, 5].)

**Theorem 3.** *If $\{s_n\}$ is summable $(l)$ to $s$, then it is summable $(L)$ to the same sum. There is a sequence summable $(L)$ but not summable $(l)$.*

**Theorem 4.** *If $\{s_n\}$ is summable $(L)$ to $s$, and if further $s_n\geq -M$, then it is summable $(l)$ to the same sum.*

These two theorems ensure the equivalence of the methods $(l)$ and $(L)$, provided that $s_n\geq -M$.

On the other hand we know the following celebrated theorems.

**Theorem 5.** *If $\{s_n\}$ is summable $(C,1)$ to $s$, then it is Abel summable to the same sum. There is a sequence Abel summable but not summable $(C,1)$.* (See [2] p. 108.)

**Theorem 6.** *If $\{s_n\}$ is Abel summable to $s$, and if further $s_n\geq -M$, then it is summable $(C,1)$ to the same sum.* (See [2] pp. 154 *et seq.*, [3, 6].)

These two theorems also ensure the equivalence of the methods $(C, 1)$ and Abel, provided that $s_n \geq -M$.

Hardy and Littlewood [3] point out that the assumption

$$f(x)=\sum_{n=0}^{\infty} p_n x^n \sim \log\left(\frac{1}{1-x}\right), \quad p_n \geq 0,$$

cannot involve

$$f'(x)=\sum_{n=1}^{\infty} n p_n x^{n-1} \sim \frac{1}{1-x}.$$

To prove it they use the power series

$$f(x)=\sum_{n=0}^{\infty} x^{a^n},$$

where $a$ is an integer greater than or equal to 2. Then we get

$$f(x) \sim \frac{1}{\log a} \log\left(\frac{1}{1-x}\right),$$

but $(1-x)f'(x)$ cannot lead to a limit as $x \to 1-0$.

If we put, in the above example, $p_0=0$ and $p_n=\frac{s_{n-1}}{n}$ for $n \geq 1$, then we get the following

**Theorem 7.** *There is a sequence* $\{s_n\}$, $s_n \geq -M$, *summable* $(L)$ *but not Abel summable.*

On the other hand we know the following

**Theorem 8.** *If* $\{s_n\}$ *is Abel summable to* $s$, *then it is summable* $(L)$ *to the same sum.* (See [1], [3] p. 81.)

On account of Theorem 1, 3, 4, 5, 6, 7, and 8 we can deduce further the following

**Theorem 9.** *There is a sequence* $\{s_n\}$, $s_n \geq -M$, *summable* $(l)$ *but not summable* $(C, 1)$.

In fact if Theorem 9 would not hold, then Theorem 7 would not hold also. Of course, we can directly prove Theorem 9 by using the sequence

$$s_{a^n-1}=a^n \quad \text{for } n=0, 1, 2, \cdots, \ a \geq 2,$$
$$s_k=0 \quad \text{for other } k.$$

But the proof is a repeat of that of Theorem 7.

## References


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