128. On Unified Representation of State Vectors in Quantum Field Theory. II

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§1. **Introduction.** The space which represents the state vectors used in quantum field theory can be seen in [1], [2], and [3].

Haag, Gårding, and Wightman have used the space $L^2(\Gamma; m)$ consisted of the square integrable functions on the set Γ of the infinite nonnegative integer's sequence with the measure m [1], [2].

For discrete measure m, it is called discrete representation, and for continuous measure m, continuous representation.

In quantum field theory, these two representations are used.

We can easily show the necessity of this continuous representation using the model by Van Hove and O. Miyatake [4], [5], [6] which is the scalar field with the interaction by the fixed point source.

Von Neumann has also constructed the infinite direct product space to represent the state vectors which contain the spaces $L^2(\Gamma; m)$ for any m [3].

Now we have proposed to use the new space $\Lambda^2[0,1]$ to construct the space representing the state vectors [7].

The elements in $\Lambda^2[0,1]$ are the sequences converging in $\Lambda^2[0,1]$ topology. For the classification by the uniformly equivalence, the set of these classes has the same extend as the Von Neumann's space. For the classification by the equivalence in Λ^2, the set of these classes contains discrete and continuous representations.

In this paper we select uniformly equivalent classes contained in each Λ^2 equivalent class, construct the pre-Hilbert space L and divide it in the direct sum of two representations. We can also use it for the clarification of the relation between the discrete and continuous energy momentum levels.

§2. **Some derived sequences from $\{\varphi_n\}$ in the space $\Lambda^2[0,1]$.** Let $L^2[0,1]$ denote the set of all complex-valued, measurable and square-integrable functions on the interval $[0,1]$ with the Lebesgue measure.

Let's use the abbreviations

$$I^2|f|^2(x)=\int_0^x\int_0^s |f(t)|^2\,dtds,\quad \tilde{I}^2|f|^2(x)=\int_1^x\int_1^s |f(t)|^2\,dtds.$$

In the set $L^2[0,1]$, we introduce the topology λ such that the sequence $\{\varphi_n(t)\}(\varphi_n(t)\in L^2[0,1])$ is convergent in the sense of this topology, if and only if the following eight sequences of the functions:

$$\{I^2|R_+\varphi_n|^2(x)\},\ \{\hat{I}^2|R_+\varphi_n|^2(x)\},\ \{I^2|R_-\varphi_n|^2(x)\},\ \{\hat{I}^2|R_-\varphi_n|^2(x)\},$$
$$\{I^2|\mathfrak{J}_+\varphi_n|^2(x)\},\ \{\hat{I}^2|\mathfrak{J}_+\varphi_n|^2(x)\},\ \{I^2|\mathfrak{J}_-\varphi_n|^2(x)\},\ \{\hat{I}^2|\mathfrak{J}_-\varphi_n|^2(x)\},$$

which are derived from the original $\{\varphi_n(t)\}$, are uniformly convergent on the interval $[0,1]$ [7], where $R_+\varphi_n(t)$ is the real positive part of $\varphi_n(t)$ etc.

Let $\Lambda^2[0,1]$ denote the completed space of $L^2[0,1]$ in the topology λ. Let $\{\varphi_n(t)\}$ denote the convergent sequence of the functions $\varphi_n(t)\in L^2[0,1]$ in λ topology. Let $\tilde{\Phi}=CL\{\varphi_n(t)\}$ denote a class of equivalent Cauchy sequences to $\{\varphi_n(t)\}$ in the topology λ. It is an element of the space $\Lambda^2[0,1]$ containing $\{\varphi_n(t)\}$.

Hereafter, we treat the following four sequences $\{I^2|R_+\varphi_n|^2(x)\}$, $\{I^2|R_-\varphi_n|^2(x)\}$, $\{I^2|\mathfrak{J}_+\varphi_n|^2(x)\}$, $\{I^2|\mathfrak{J}_-\varphi_n|^2(x)\}$, but we only discuss the sequence $\{I^2|R_+\varphi_n|^2(x)\}$ because we have the same results about to the other sequences.

Lemma 1. (1) *The function* $\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)$ *is right differentiable and left differentiable everywhere in the interval* $[0,1]$.

(2) *The functions* $D^+\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)$ *and* $D^-\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)$ *are positive increasing.*

(3) $D^+\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)\geqq D^-\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)$ *for all values of* x *in the interval* $[0,1]$.

(4) $D^-\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x_2)\geqq D^+\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x_1)$ *for any pair* x_1, x_2 *such that* $0\leqq x_1<x_2\leqq 1$.

(5) $D\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)$ *is differentiable almost everywhere in the interval* $[0,1]$, *and is positive increasing.*

Since $\lim_{n\to\infty} I^2(|R_+\varphi_n|^2(x))$ is convex, Lemma 1 can be proved.

This lemma will be valid to define the discontinuous sum $D(x; R_+\varphi_n)$ of the function $D^+(\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x))$.

Definition 1. *We denote by* $D(x; R_+\varphi_n)$ *the function whose value for* x *is the sum of the leaps of the function* $D^+(\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x))$ *in the interval* $[0,x]$.

Definition 2. *We denote by* $CI^2|R_+\varphi_n|^2$, $DI^2|R_+\varphi_n|^2$, *the following*:

$$CI^2|R_+\varphi_n|^2=\int_0^x \{D^+\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)-D(x; R_+\varphi_n)\}\,dx$$

$$DI^2|R_+\varphi_n|^2=\int_0^x D(x; R_+\varphi_n)\,dx.$$

§3. The construction of the space L. The space $\Lambda^2[0,1]$ is not a topological linear space (III in [8]). In this paragraph we show that it contains a linear manifold L.

Lemma 2. *The functions* $CI^2|R_+\varphi_n|^2$, $DI^2|R_+\varphi_n|^2$ *do not depend on the choice of the sequence* $\{\varphi_n(t)\}$ *from the class* $\tilde{\Phi}=CL\{\varphi_n(t)\}$. *In the other words,* $CI^2|R_+\varphi_n|^2=CI^2|R_+\psi_n|^2$, $DI^2|R_+\varphi_n|^2=DI^2|R_+\psi_n|^2$, *for any pair* $\{\varphi_n(t)\}$, $\{\psi_n(t)\}$ *of the same class* $\tilde{\Phi}$.

Proof. Since $\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)=\lim_{n\to\infty} I^2|R_+\psi_n|^2(x)$ for any pair $\{\varphi_n(t)\}$, $\{\psi_n(t)\}$ in the same class $\tilde{\Phi}$, it follows that $D(x, R_+\varphi_n)=D(x, R_+\psi_n)$. So $CI^2|R_+\varphi_n|^2=CI^2|R_+\psi_n|^2$ and $DI^2|R_+\varphi_n|^2=DI^2|R_+\psi_n|^2$.

Definition 3. $CI^2|R_+\tilde{\Phi}|^2=CI^2|R_+\varphi_n|^2$, $DI^2|R_+\tilde{\Phi}|^2=DI^2|R_+\varphi_n|^2$.

Definition 4. $I^2|R_+\tilde{\Phi}|^2=\lim_{n\to\infty} I^2|R_+\varphi_n|^2(x)$.

Definition 5. $D(x; R_+\tilde{\Phi})=D(x; R_+\varphi_n)$.

Lemma 3. (1) $\infty>\{D^+(I^2|R_+\tilde{\Phi}|^2)\}_{1+0}\geqq D(1; R_+\tilde{\Phi})\geqq 0$.

(2) $d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)\geqq 0$ *in almost everywhere and* $\sqrt{d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)}$ *is contained in the space* $L^2[0,1]$.

(3) $d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)$ *is a continuous function on the interval* $[0,1]$ *and is divided into the following two parts*

$$d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)=\int_0^x d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)\,dx+\lim_{n\to\infty}\int_0^x \sum_i A_i^{(n)}\delta(x_i)\,dx,$$

where x_i *is a point in the set* $\{x; D^+d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)=+\infty$ *or* $D^-d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)=+\infty\}$, $A_i^{(n)}\geqq 0$, $\lim_{n\to\infty} A_i^{(n)}=0$ *and* $\lim_{n\to\infty}\sum_i A_i^{(n)}<+\infty$.

(4) $D^+\{D(x; R_+\tilde{\Phi})\}=\sum_{k=1}^{\infty} C_k^2\delta(x_k)$ *where* $0\leqq x_k\leqq 1$ *and* C_k *are real numbers such that* $\sum_{k=1}^{\infty} C_k^2<+\infty$.

Proof. From the Definitions 1 and 2, we can see that the function $d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)$ is continuous. Then (1) is easily obtained.

Since the function $d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)$ is monotone increasing and continuous on the interval $[0,1]$, its derivative $d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)$ is defined almost everywhere in the interval $[0,1]$. The derivative $d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)$ has the following three properties:

(a) $d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)>0$.

(b) $+\infty>d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)\geqq\int_0^x d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)\,dx\geqq 0$.

(c) $\{d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)-\int_0^x d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)\,dx\}$ is a monotone increasing continuous function. (a) and (b) prove the property (2).

Since the strictly increasing point of the function $\left\{d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)-\int_0^x d^2/dx^2\, CI^2|R_+\tilde{\Phi}|^2(x)\,dx\right\}$ belongs to the set $\{x; D^+d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)=+\infty$ or $D^-d/dx\, CI^2|R_+\tilde{\Phi}|^2(x)=+\infty\}$, the function is expressed in the form $\lim_{n\to\infty}\int(\sum_i C_i^{(n)}\delta(x_i))\,dx$, and the coefficients $C_i^{(n)}$ satisfy the conditions $C_i^{(n)}\geqq 0$, $\lim_{n\to\infty} C_i^{(n)}=0$ and $\lim_{n\to\infty}\sum_i C_i^{(n)}<+\infty$. This prove the property (3).

According to the Definitions 1, 5, $D^+\{D(x;R_+\tilde{\Phi})\}=\sum_{k=1}^{\infty}C_k^2\delta(x_k)$ and $+\infty>[D^+I^2|R_+\tilde{\Phi}|^2]_{1+0}\geqq\sum_{k=1}^{\infty}C_k^2$. This proves the property (4).

Now we can derive the following

Definition 6. $\tilde{D}(x;R_+\tilde{\Phi})=D(x;R_+\tilde{\Phi})+d/dx\,CI^2|R_+\tilde{\Phi}|^2(x)-$ $-\int_0^x d^2/dx^2\,CI^2|R_+\tilde{\Phi}|^2(x)\,dx.$

We construct the linear manifold L such that for any class $\tilde{\Phi}$ of $\Lambda^2[0,1]$ there exists some element Φ of L which is the subset of $\tilde{\Phi}$.

Let us select a sequence $\{\varphi_n^0\}$ from $\tilde{\Phi}$ for this purpose.

At the first step, let $\rho_{x_0n}(x)$ be the following functions:

$$\rho_{x_0n}(x)=\begin{cases}\rho_{1/n^2}(x-x_0) & \text{for } 0<x_0<1\\ 2\rho_{1/n^2}(x-x_0) & \text{for } x=0,1,\end{cases}$$

where $0\leqq x\leqq 1$, and $\rho_\varepsilon(x)$ is the function defined by L. Schwartz.

At the second step we define the sequence $\{f_{\tilde{D},R_+\tilde{\Phi}}^{(m)}(x)\}$ of the following functions:

$$f_{\tilde{D},R_+\tilde{\Phi}}^{(m)}(x)=\sum_{k=1}^{2^m}\sqrt{\tilde{D}(k/2^m+0;R_+\tilde{\Phi})-\tilde{D}((k-1)/2^m-0;R_+\tilde{\Phi})}\sqrt{\rho_{k/2^m,\,4^m+10}(x)},$$

where $\tilde{D}(x-0;R_+\tilde{\Phi})=0$ for $x<0$. Similarly we construct the sequences $\{f_{\tilde{D},R_-\tilde{\Phi}}^{(m)}(x)\}$, $\{f_{\tilde{D},\mathfrak{J}_+\tilde{\Phi}}^{(m)}(x)\}$, and $\{f_{\tilde{D},\mathfrak{J}_-\tilde{\Phi}}^{(m)}(x)\}$.

At the third step we define the sequence $\{f_{C,R_+\tilde{\Phi}}^{(m)}(x)\}$ from the following functions: $f_{C,R_+\tilde{\Phi}}^{(m)}(x)=\sqrt{d^2/dx^2\,CI^2|R_+\tilde{\Phi}|^2(x)}$. Similarly we define the sequences $\{f_{C,R_-\tilde{\Phi}}^{(m)}(x)\}$, $\{f_{C,\mathfrak{J}_+\tilde{\Phi}}^{(m)}(x)\}$, and $\{f_{C,\mathfrak{J}_-\tilde{\Phi}}^{(m)}(x)\}$. The sequence $\{f_{C,R_+\tilde{\Phi}}^{(m)}(x)-f_{C,R_-\tilde{\Phi}}^{(m)}(x)+if_{C,\mathfrak{J}_+\tilde{\Phi}}^{(m)}(x)-if_{C,\mathfrak{J}_-\tilde{\Phi}}^{(m)}(x)+f_{\tilde{D},R_+\tilde{\Phi}}^{(m)}(x)$ $-f_{\tilde{D},R_-\tilde{\Phi}}^{(m)}(x)+if_{\tilde{D},\mathfrak{J}_+\tilde{\Phi}}^{(m)}(x)-if_{\tilde{D},\mathfrak{J}_-\tilde{\Phi}}^{(m)}(x)\}$ is contained in Φ. We denote this sequence by $\{\varphi_m^0\}$. Let Φ denote an uniformly equivalent class of $\{\varphi_n^0\}$ [7].

Definition 7. *Let L denote the aggregate of all above Φ.*

Lemma 4. *The sequences $\{\varphi_n^0\}$ selected from the class $\tilde{\Phi}\in\Lambda^2[0,1]$ have the following properties:*

(**1**) *φ_n^0 is decomposed in the following form:*

$$\varphi_n^0(x)=g_1(x)-g_2(x)+ig_3(x)-ig_4(x)$$
$$+\textstyle\sum_k(C_{k1}^{(n)}-C_{k2}^{(n)}+iC_{k3}^{(n)}-iC_{k4}^{(n)})\sqrt{\rho_{k/2^n,\,4^n+10}(x)}$$
$$(n=1,2,\cdots,\ k=1,\cdots 2^n),$$

where $g_i(x)$, for $i=1,2,3,4$ are non-negative functions in $L^2[0,1]$.

$C_{ki}^{(n)}\geqq 0$ and $|\lim_{n\to\infty}\sum_{k<A2^n}(C_{ki}^{(n)})^2|<+\infty$ for $i=1,2,3,4$ and for all A such that $0<A<1$, and

(**2**) *the carrier of the generalized function $\{\lim_{n\to\infty}\sum_k C_{ki}^{(n)}\sqrt{\rho_{k/2^n,\,4^n+10}(x)}\}$*

is contained in the set $\{x; |g_i(x)|<+\infty\}^C$ *for* $i=1, 2, 3, 4$ *or an arbitrary countable point set.*

Conversely, if the sequence $\{\varphi_n^0\}$ *satisfies the conditions* (1), (2), *then there exists a* $\tilde{\Phi}\in \Lambda^2[0, 1]$ *such that* $\{\varphi_n^0\}$ *is selected as the representative of an element* $\tilde{\Phi}\in\Lambda^2[0, 1]$ *obeying the above process.*

Theorem 1. *The space* L *is a linear manifold.*

Proof. Suppose that the sequences $\{\varphi_{n1}^0\}$, $\{\varphi_{n2}^0\}$ are selected from $\tilde{\Phi}_1, \tilde{\Phi}_2\in\Lambda^2[0, 1]$.

Since the inequalities

$$|R_+(\varphi_{n1}^0+\varphi_{n2}^0)|^2\leqq|R_+\varphi_{n1}^0+R_+\varphi_{n2}^0|^2\leqq 2\{|R_+\varphi_{n1}^0|^2+|R_+\varphi_{n2}^0|^2\}$$

$|R_+(\alpha\varphi_{n1}^0)|^2\leqq|\alpha|^2|R_+\varphi_{n1}^0|^2$ etc. hold, and the sequences $\{\varphi_{n1}^0+\varphi_{n2}^0\}$ and $\{\alpha\varphi_{n1}^0\}$ have the properties (1) and (2) in Lemma 4, there exist the classes in $\Lambda^2[0, 1]$ which contain the sequences $\{\varphi_{n1}^0+\varphi_{n2}^0\}$, $\{\alpha\varphi_{n1}^0\}$.

So L is a linear manifold.

Definition 8. *A set* M *in the space* $\Lambda^2[0, 1]$ *is called the mother set in* $\Lambda^2[0, 1]$, *if*

(i) *any element* Φ *in* M *contains an above sequence* $\{\varphi_n^0\}$.

(ii) *any above sequence* $\{\varphi_n^0\}$ *is contained in an element* $\Phi\in M$.

The correspondence between the space M and the space L is one-to-one. But the following example show that the correspondence between the space M and the space $\Lambda^2[0, 1]$ is not one-to-one.

Example. *Let the sequences* $\{\varphi_n^{(1)}\}$, $\{\varphi_n^{(2)}\}$ *consist of the following functions' elements*:

$$\varphi_n^{(1)}=\sum_{k=1}^{2^n}\sqrt{\rho_{k/2^n,\, 4^n+10}(x)}\,/\sqrt{2^n}$$

$$\varphi_n^{(2)}=\sum_{k=1}^{2^n}\sqrt{2\rho_{k/2^n,\, 4^n+10}(x)}\,/\sqrt{2^n}-\sum_{k=1}^{2^n}\sqrt{\rho_{k/2^n-1/2^{n+1},\, 4^n+10}(x)}\,/\sqrt{2^n}.$$

In the space $\Lambda^2[0, 1]$, $\{\varphi_n^{(1)}\}\neq\{\varphi_n^{(2)}\}$ *but in the space* L $\{\varphi_n^{(1)}\}=\{\varphi_n^{(2)}\}$.

§4. Unified Representation. Since for any pair of sequences $\{\varphi_{n1}^0(t)\}$, $\{\psi_{n1}^0(t)\}$ in $\Phi_1\in L$ and $\{\varphi_{n1}^0(t)\}$, $\{\psi_{n2}^0(t)\}$ in $\Phi_2\in L$

$$\lim_{n\to\infty}\langle\varphi_{n1}^0(t),\,\varphi_{n2}^0(t)\rangle=\lim_{n\to\infty}\langle\varphi_{n1}^0(t),\,\psi_{n2}^0(t)\rangle$$
$$=\lim_{n\to\infty}\langle\psi_{n1}^0(t),\,\varphi_{n2}^0(t)\rangle=\lim_{n\to\infty}\langle\psi_{n1}^0(t),\,\psi_{n2}^0(t)\rangle.$$

We can define the inner product between the element Φ_1 and Φ_2 in L by $\langle\Phi_1, \Phi_2\rangle=\langle\{\varphi_{n1}^0(t), \{\varphi_{n2}^0(t)\}\rangle=\lim_{n\to\infty}\langle\varphi_{n1}^0(t), \varphi_{n2}^0(t)\rangle$. We denote $\mathfrak{L}$ the space L with this inner product.

Let S denote the set constructed from the sequences

$$\{\sum_k(C_{k1}^{(n)}-C_{k2}^{(n)}+iC_{k3}^{(n)}-iC_{k4}^{(n)})\sqrt{\rho_{k/2^n,\, 4^n+10}(x)}\}$$

with the properties (1) and (2) in Lemma 4.

Let Σ denote the set S with the topology introduced from $\mathfrak{L}$.

From the one-to-one correspondence between the convergent sequences $\{k/2^m\}$ $(m=1,2,\cdots, k=1,2,\cdots, 2^m)$ and the points in the interval $[0, 1]$, we obtain the following

Lemma 5. *The pre-Hilbert space Σ contains a non-separable Hilbert space such that its bases are $\sqrt{\delta(x)}$ $(0 \leqq x \leqq 1)$ as a component of orthogonal direct sum. (It contains a discrete respresentation.)*

Now we see the following

Theorem 2. *$\mathfrak{L}$ is the direct sum of the two spaces*

$$\mathfrak{L} = L^2[0, 1] \oplus \Sigma.$$

Proof. Using the Schwartz inequality, the orthogonal relation $\langle \{f_{C, R+\tilde{\Psi}}(x) - f_{C, R-\tilde{\Psi}}(x) + if_{C, \mathfrak{J}+\tilde{\Psi}}(x) - if_{C, \mathfrak{J}-\tilde{\Psi}}(x)\}, \{f_{\tilde{D}, R+\tilde{\Phi}}(x) - f_{\tilde{D}. R-\tilde{\Phi}}(x) + if_{\tilde{D}, \mathfrak{J}+\tilde{\Phi}}(x) - if_{\tilde{D}, \mathfrak{J}-\tilde{\Phi}}(x)\} \rangle = 0$ holds for all Ψ and Φ in L.

From Lemma 4 $\{f_{C, R+\tilde{\Psi}}(x) - f_{C, R+\tilde{\Psi}}(x) + if_{C, \mathfrak{J}+\tilde{\Psi}}(x) - if_{C, \mathfrak{J}-\tilde{\Psi}}(x)\}$ construct the space $L^2[0, 1]$.

Here $L^2[0, 1]$ correspond to a continuous representation.

Hence $\mathfrak{L}$ is decomposed in the following form: $\mathfrak{L} = L^2[0, 1] \oplus \Sigma$ and Σ has the properties in Lemma 5.

(This article is dedicated to professor Kunugi on the occasion of his 60th birthday.)

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